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THE

PARABOLA, ELLIPSE, AND HYPERBOLA,

TREATED GEOMETRICALLY

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THE

PARABOLA, ELLIPSE, AND HYPERBOLA,

TREATED GEOMETRICALLY.

BY

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PREFACE.

LONG experience has proved to me that it would be a great practical advantage for the general class of students to acquire a knowledge of even the elementary Geometrical properties of the Parabola, Ellipse, and Hyperbola—a knowledge which may be attained, in a very short time, by any one acquainted with the six Books of Euclid, though many have neither the ability nor the perseverance necessary for the Analytical investigation.

In the following Treatise I have endeavoured to demonstrate, on strictly geometrical principles, the most useful properties of these curves. In the definition of a tangent I have avoided the notion of a limit, adopting, in preference, Euclid's definition; and the demonstrations depending thereon will, I think, be found somewhat new and interesting.

The similar properties of the three curves will be found to be treated in such a manner as to need scarcely any change of either words or notation.

In Chap. III. Proposition XIX. *et seq.* and Corollaries, I have traced the analogy between the properties of conjugate diameters of the Hyperbola, and those of the Ellipse, further than has been done in any other Treatise.

To the Board of Trinity College I owe my grateful thanks, for extending to me that liberal support with which they have ever shown themselves ready to assist the humblest efforts.

<div align="right">ROBERT WM. GRIFFIN.</div>

19, TRINITY COLLEGE,
 September, 1879.

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CHAPTER I.

The Parabola.

DEFINITIONS.

A Parabola is the curve traced out by a point, which moves in such a way that its distance from a fixed point is always equal to its perpendicular distance from a fixed right line.

The fixed point is called the *Focus*, and the fixed right line the *Directrix*.

Any right line perpendicular to the Directrix is called a *Diameter*.

The right line drawn through the Focus perpendicular to the directrix is called the *Axis*, and the point at which it meets the curve the *Vertex*.

A right line which meets the curve, and, being produced, does not cut it, is called a *Tangent*.

A right line drawn through any point on the curve perpendicular to the tangent at that point is called a *Normal*.

If a right line be drawn parallel to any tangent, the part intercepted on it, between the curve and a diameter passing through the point of contact of the tangent, is called an *Ordinate* to that diameter.

The part intercepted on any diameter between the ordinate and the curve is called the *Abscissa*.

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The right line joining any two points on the curve is called a *Chord*.

The chord drawn through the Focus parallel to the ordinates to any diameter is called the *Parameter* of that diameter.

The chord drawn through the focus at right angles to the axis is called the *Latus rectum*.

PROPOSITION I.

The focus and directrix of a parabola being given, to determine any number of points on the curve.
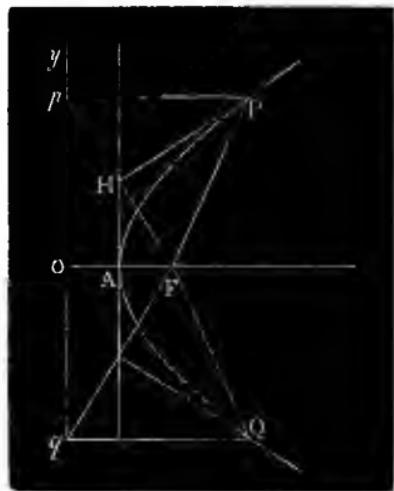
Fig. 1.

Let F be the focus, and Oy the directrix.

Draw $FO \perp$ the directrix; bisect FO at A.

Then OF is the axis, and A the vertex of the curve.

On the directrix take any point p; join Fp; draw $AH \perp FO$; through H draw $HP \perp Fp$, to meet pP drawn $\perp$ the directrix; join PF.

Then $\triangle^s FHP$ and pHP are equal; (4 I. Euclid.)

∴ $PF = Pp$. Hence P is a point on the curve.

In like manner, by joining the focus with other points on the directrix, any number of points on the curve may be determined.

Cor. 1.—If Oq be taken $= Op$, another point Q may be found in a similar manner to P, which will be at the same distance both from the axis and the directrix ; hence the curve is symmetrical with regard to the axis.

Cor. 2.—Since $Pp = Qq$;

$$\therefore \quad FP = FQ, \text{ also } \angle OFP = \angle OFQ ;$$

$\therefore$ right lines drawn from the focus to the curve, making equal angles with the axis, are equal.

Cor. 3.—The Latus rectum is equal to four times the distance of the focus from the vertex.

Fig. 2.

Draw PP' through the focus $\perp$ the axis ; and $Pp \perp$ the directrix.

Then $\qquad\qquad PP' = 2PF \qquad\qquad$ (*Cor.* 2.)

$$= 2Pp$$

$$= 2FO = 4FA.$$

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Cor. 4.—If any parallel qR be drawn to the axis between the vertex A and any point P on the curve, the segment qQ intercepted on this line between the curve and the directrix is less than the segment qR intercepted on it between the line AP and the directrix; and, therefore, the curve is convex towards the directrix.
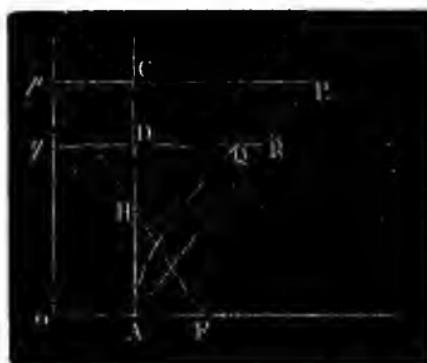

Fig. 3.

Since $\angle FHP$ is a right $\angle$; $\therefore \angle AHF + \angle CHP = 90°$.

Hence $\quad\quad \triangle AHF$ is similar to $\triangle CPH$;

$\quad \therefore FA : AH = HC : CP$;

$\quad \therefore FA \times CP = AH \times HC.$

But $\quad FA = AO$; $\therefore FH = Hp$; $\therefore AH = HC$;

$\quad \therefore FA \times CP = AH^2 = \tfrac{1}{4}AC^2.$

Similarly, $\quad\quad FA \times DQ = \tfrac{1}{4}AD^2$;

$\quad \therefore CP : DQ = AC^2 : AD^2$

$\quad\quad\quad\quad\quad = CP^2 : DR^2$; $\quad\quad$ (Similar $\triangle^s$.)

$\quad \therefore CP^2 : CP \times DQ = CP^2 : DR^2$;

$\quad \therefore CP \times DQ = RD^2$;

$\quad \therefore DQ : RD = RD : CP = AD : AC.$ (Similar $\triangle^s$.)

But by hyp., $\quad AD < AC$; $\therefore DQ < DR$; $\therefore qQ < qR.$

Cor. 5.—Also if a perpendicular QN be drawn to the axis between the vertex A and any point P on the curve, the segment QN intercepted on this line between the curve and the axis is greater than the segment SN intercepted on it between the line AP and the axis; and, therefore, the curve is concave towards the axis.
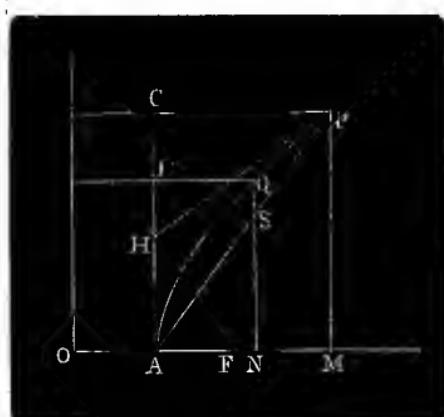
Fig. 4.

For

$$CP : DQ = AC^2 : AD^2 ; \qquad (Cor.\ 4.)$$

$$\therefore\ AM : AN = PM^2 : QN^2 ;$$

$$\therefore\ PM : SN = PM^2 : QN^2 ; \qquad (\text{Similar } \triangle^s.)$$

$$\therefore\ PM^2 : PM \times SN = PM^2 : QN^2 ;$$

$$\therefore\ PM \times SN = QN^2 ;$$

$$\therefore\ PM : QN = QN : SN ;$$

but it was proved above that

$$PM^2 : QN^2 = AM : AN,$$

and by hyp.,

$$AN < AM ; \ \therefore\ QN < PM, \text{ and } \therefore\ SN < QN.$$

PROPOSITION II.

The distance of any point from the focus is greater or less than its distance from the directrix according as the point is on the convex or concave side of the curve.
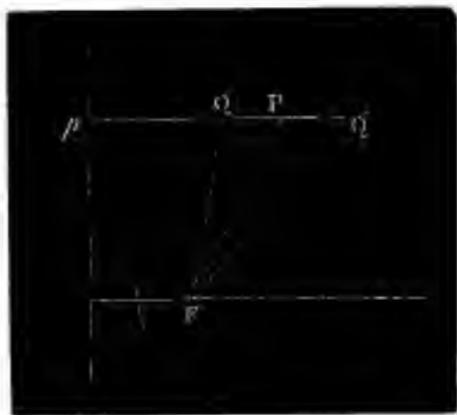

Fig. 5.

First let the point Q be on the convex side; join QF, and let the perpendicular Qp on the directrix be produced to meet the curve in P; join PF.

Then $\qquad QF + QP > PF;$ \qquad (20 I. Euclid.)

$\therefore \ QF + QP > Pp\,;$

$\therefore \qquad QF > Qp.$

Next let Q' be on the concave side; join $Q'F$.

Then $\qquad Q'F < Q'P + PF;$ \qquad (20 I. Euclid.)

$\therefore \ Q'F < Q'P + Pp\,;$

$\therefore \ Q'F < Q'p.$

Cor.—Conversely a point will be on the concave or convex side of a parabola according as its distance from the focus is less or greater than its perpendicular distance from the directrix.

PROPOSITION III.

1°. The line which bisects the angle between that drawn from any point on a parabola to the focus, and that drawn perpendicular to the directrix, falls wholly without the curve.

2°. Any other line drawn through the point will cut the curve.
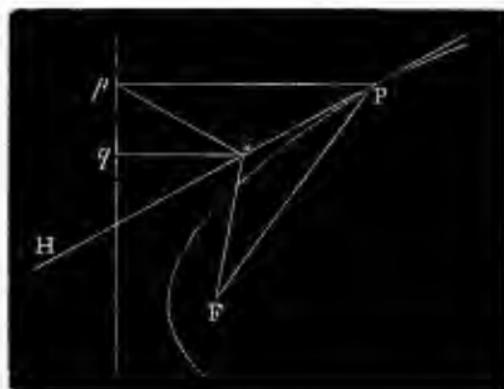
Fig. 6.

1°. Let PH bisect the angle between PF and Pp, the perpendicular on the directrix.   Take any point Q on PH; join QF, and draw $Qq \perp$ the directrix.

Then the $\triangle^s QPF$ and QPp are equal;   (4 I. Euclid.)

$$\therefore QF = Qp,$$

but $$Qp > Qq;$$ (19 I. Euclid.)

$$\therefore QF > Qq;$$

$\therefore Q$ is a point without the curve.   (*Cor.* Prop. II.)

2°. Let *PK* be any line other than the bisector of the
∠ *FPp*.
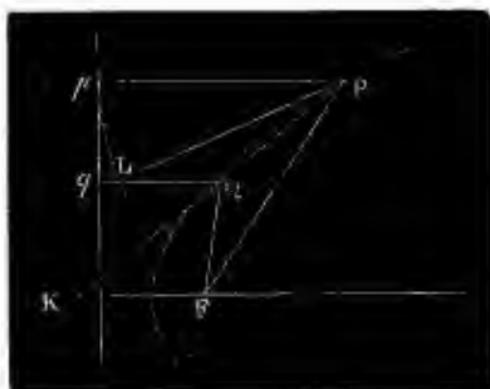
Fig. 7.

Draw *PL*, making the ∠ *KPL* = ∠ *KPF*; cut off
PL = *PF*. Through *L* draw *QLq* ⊥ the directrix; join
QF.

Then the △ˢ *FPQ* and *LPQ* are equal;   (4 I. Euclid.)

$$\therefore QL = QF;$$

$$\therefore QF < Qq;$$

∴ the point *Q* is on the concave side of the curve.

(*Cor.* Prop. II.)

Cor. 1.—Hence the line which bisects the angle be-
tween that drawn from any point on the curve to the
focus, and that drawn perpendicular to the directrix, is a
Tangent to the curve.

Cor. 2.—The tangent at the vertex is perpendicular to
the axis, and ∴ ordinates to the axis are also perpendi-
cular to it.

Cor. 3.—The normal to a parabola at any point bisects
the external angle between the line drawn from that point
to the focus, and the perpendicular let fall on the directrix.

PROPOSITION IV.

The locus of the foot of the perpendicular let fall from the focus on any tangent to a parabola is the tangent at the vertex.
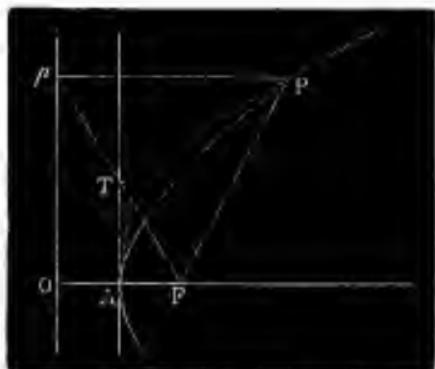

Fig. 8.

Let PT be any tangent, and FT the perpendicular let fall from the focus. Draw $Pp \perp$ the directrix; join pT.

Then the $\triangle^s FPT$ and pPT are equal;   (4 I. Euclid.)

$$\therefore \angle pTP = \angle FTP = 90°;$$

$\therefore$ Tp and TF are in directum.   (14 I. Euclid.)

Also $Tp = TF$; $\therefore$ TA is $\parallel Op$.   (2 VI. Euclid).

Hence TA is a tangent at the vertex.

(Cor. 2, Prop. III.)

Cor. 1.—If the vertex of a right $\angle FTP$ move along a fixed right line, while one leg passes through a fixed point F, the other leg will always touch a parabola, of which the fixed point is the focus, and the fixed right line the tangent at the vertex.

Cor. 2.—If a perpendicular Pp be let fall on the directrix from any point P on a parabola, the line Fp joining the foot of this perpendicular with the focus is $\perp$ the tangent at the point P.

Cor. 3.—By the aid of this Prop. we can draw a pair of tangents to a parabola from any point Q without it.
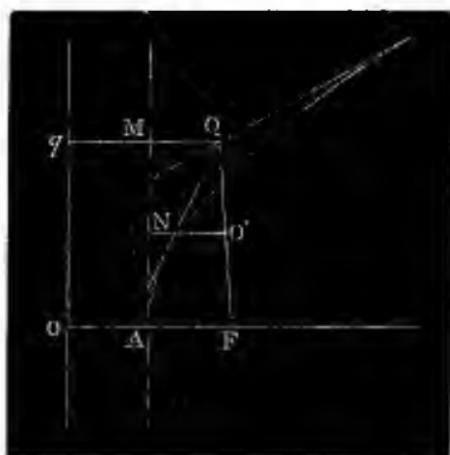

Fig. 9.

On the line joining Q with the focus as diameter describe a circle; the line joining the given point Q with the points of intersection of this circle with the tangent at the vertex will be the required tangents.

Note.—The circle described on QF as diameter will always intersect the tangent at the vertex.

For, bisect QF in O' and draw $O'N \perp AM$.

Then $Qq = QM + Mq = QM + AO = QM + AF = 2O'N,$

but $QF > Qq$ (Prop. II.); $\therefore QF > 2O'N$ or $O'F > O'N.$

That is, the radius of the circle is always greater than the perpendicular let fall from its centre on the tangent at the vertex.

PROPOSITION V.

Any line drawn through the focus is cut harmonically by the curve, the focus, and the directrix.
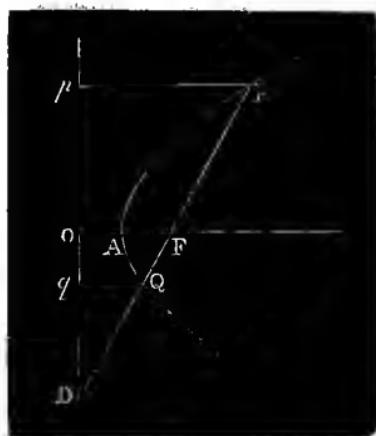
Fig. 10.

For $\qquad PD : DQ = Pp : Qq \qquad$ (Similar $\triangle^s$.)

$$= PF : FQ.$$

Hence $\;\; DP, DF,$ and DQ are in harmonic proportion.

Cor. 1. $\qquad Pp : FO : Qq = DP : DF : DQ.$

(Similar $\triangle^s$.)

Hence $\;\; Pp, FO,$ and $Qq,$ are in harmonic proportion.

$\therefore \;\; \dfrac{1}{Pp}, \dfrac{1}{FO},$ and $\dfrac{1}{Qq},$ are in Arith. Progression.

$\therefore \;\; \dfrac{1}{Pp} + \dfrac{1}{Qq} = \dfrac{2}{FO};$ and $\therefore$ constant.

Cor. 2.—Hence, also, $\dfrac{1}{FP} + \dfrac{1}{FQ}$ is constant.

PROPOSITION VI.

The locus of the intersection of tangents to a parabola which cut at right angles is the directrix.
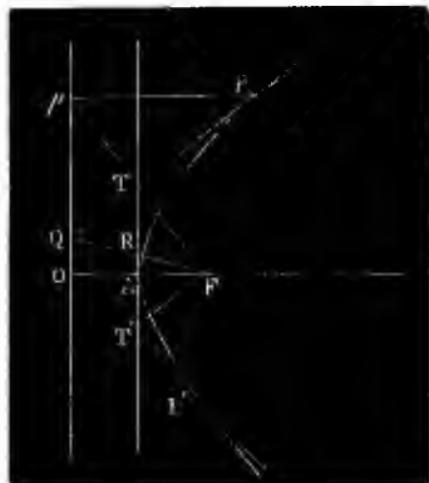
Fig. 11.

Let QP, QP', be tangents which cut at right angles. From the focus draw the perpendiculars FT, FT', on the tangents.

Then T and T' are points on the tangents at the vertex.

(Prop. IV.)

$\therefore$ TT' is $\perp$ the axis.   (*Cor.* 2, Prop. III.)

Also since $FTQT'$ is a rectangle, its diagonal QF is bisected at R.

Draw $QO \parallel TT'$.

Then $FA = AO$;   (2 VI. Euclid.)

and QO being $\parallel TT'$, is $\therefore$ $\perp$ the axis.

Hence $\therefore$ QO is the directrix.

PROPOSITION VII.

If PN be a normal to a parabola at any point P, and PM the ordinate to the axis, the subnormal MN is of a constant length and equal to half the latus rectum.
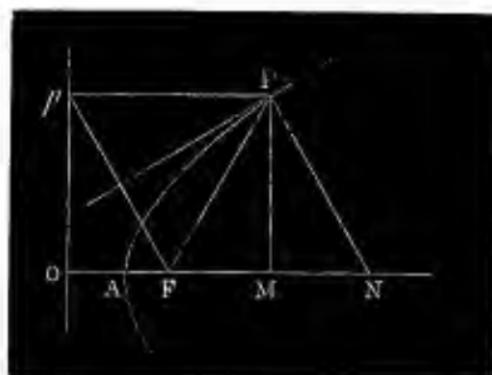
Fig. 12.

Draw $Pp \perp$ the directrix.

Then Fp will be $\perp$ the tangent at P; and $\therefore \parallel PN$.
(Cor. 2, Prop. IV.)

Hence $\triangle FpO = \triangle NPM$;   (26 I. Euclid.)

$\therefore MN = FO = 2FA = \frac{1}{2}$ Latus rectum.
(Cor. 3, Prop. I.)

Def.—The portion of the axis intercepted between the normal at any point on the curve, and the ordinate to the axis drawn through the same point, is called the *Subnormal*.

The portion of the axis intercepted between the tangent at any point on the curve, and the ordinate to the axis drawn through the same point, is called the *Subtangent*.

PROPOSITION VIII.

If a tangent at any point P on a parabola meet the axis produced in T, and the ordinate PM be drawn, then the subtangent MT will be bisected at the vertex.
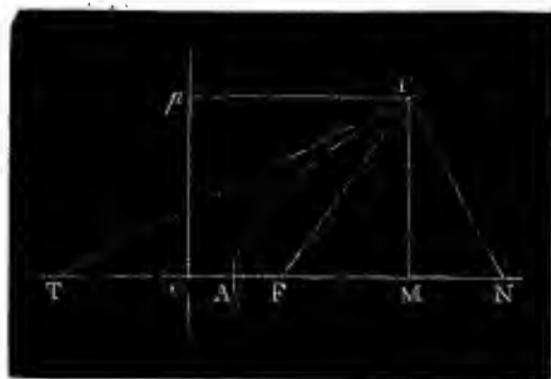
Fig. 13.

Draw $Pp \perp$ the directrix ; join FP.

Then $\angle FPT = \angle TPp = \angle PTF$;

$\therefore FT = FP = Pp = MO$;

but $AF = AO$;

$\therefore AT = AM.$

Cor. 1.—If PN be a normal, then $FT = FN.$

For $FT = MO = MF + FO = MF + MN = FN.$

(Prop. VII.)

Cor. 2. $\angle PFN = 2 \angle PTF.$

PROPOSITION IX.

The rectangle under the Latus rectum and the abscissa is equal to the square of the ordinate.
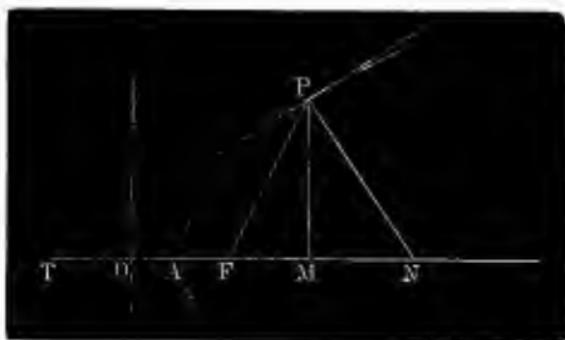
Fig. 14.

Draw PT the tangent and PN the normal at the point P.

Then $\quad PM^2 = TM \times MN$; $\qquad$ (8 VI. Euclid.)

but $\qquad TM = 2AM$; and $MN = 2AF$;

$\qquad\qquad\qquad\qquad$ (Props. VII. and VIII.)

$\therefore PM^2 = 2AM \times 2AF = 4AF \times AM.$

Cor. 1.—The squares of the ordinates to the axis are proportional to the abscissas.

Cor. 2.—The square of the tangent $= 4FP \times AM.$

For $\qquad\qquad TP^2 = TN \times TM \qquad$ (8 VI. Euclid.)

$\qquad\qquad\qquad = 2TF \times 2AM$

$\qquad\qquad\qquad = 4FP \times AM.$

Cor. 3.—The square of the normal $= 4AF \times FN.$

For $PN^2 = TN \times NM = 2FN \times 2AF = 4AF \times FN.$

PROPOSITION X.

The locus of the intersection of any tangent to a para-
bola with the line drawn through the focus perpendicular
to the radius vector drawn from the focus to the point of
contact of the tangent is the directrix.

Fig. 15.

Let PT be the tangent and FT the $\perp$ the radius vector
FP.

Draw $TO \perp$ the axis, and $Pp \perp TO$ produced.

Then $\triangle TFP = \triangle TpP$; (26 I. Euclid.)

$$\therefore Pp = PF;$$

hence OT is the directrix.

Cor. 1.—Conversely, if from any point on the directrix
a tangent be drawn to a parabola, the line joining that
point to the focus is perpendicular to the radius vector
drawn from the focus to the point of contact of the
tangent.

Cor. 2.—Tangents at the extremities of a focal chord
intersect at right angles on the directrix.

For $\angle FTP = \angle pTP$ and $\angle FTQ = \angle qTQ$;

$$\therefore QTP = \tfrac{1}{2}(\angle FTp + \angle FTq)$$

$$= \text{a right } \angle.$$

Proposition XI.

If any chord PP' of a parabola cut the directrix in D, then FD is the external bisector of the angle PFP'.
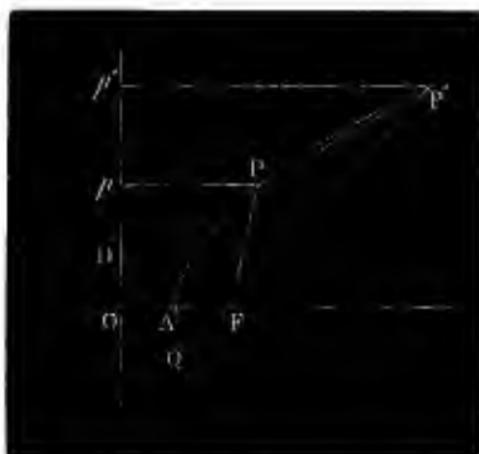
Fig. 16

Draw Pp, $P'p'$ $\perp$ the directrix.

Then $\qquad P'F : PF = P'p' : Pp$

$\qquad\qquad\qquad = P'D : DP;$ \qquad (Similar Δ^s.)

$\qquad \therefore FD$ is the bisector of the $\angle PFQ$.

(3 VI. Euclid.)

Cor. 1.—Hence, being given the focus F and two points P, P', on a parabola, we can find the directrix and axis.

For, draw FD bisecting the $\angle$ between FP and FP' produced; the point D where this line meets PP' produced is a point on the directrix. The tangent drawn from D to the circle described with the centre P and the radius PF, will be the directrix, and the perpendicular let fall on it from the focus, the axis.

If a tangent to a curve be defined as *the line joining two indefinitely near points on the curve*, it will follow immediately from this Prop. that the right line drawn from the focus to the point of intersection of any tangent with the

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directrix is perpendicular to the radius vector drawn from the focus to the point of contact of the tangent; also, that the tangent makes equal angles with the lines drawn from the point of contact to the focus and the perpendicular to the directrix.

For when P is indefinitely near to P', PD becomes a tangent at the point P, and the $\angle PFP'$ is indefinitely small, and $\therefore PFQ = 180°$; but $\angle PFD = \frac{1}{2} \angle PFQ$, $\therefore \angle PFD = 90°$.

Also, $\triangle PFD = \triangle PpD$;

$$\therefore \angle DPF = \angle DPp.$$

PROPOSITION XII.

If two fixed points PP' on a parabola be joined with a third variable point O on the parabola, the segment pp' intercepted on the directrix by the produced chords subtends a constant angle at the focus.


Fig. 17.

Since (Prop. XI.) Fp is the bisector of the $\angle$ between FO and FP produced;

$$\therefore \angle OFp + \frac{1}{2} \angle OFP = 90°;$$

also, since (Prop. XI.) Fp' is the bisector of $\angle$ between FO and FP' produced;

$$\therefore \angle OFp' + \tfrac{1}{2} \angle OFP' = 90° ;$$
$$\therefore \angle OFp + \tfrac{1}{2} OFP = \angle OFp' + \tfrac{1}{2} \angle OFP' ;$$
$$\therefore \angle OFp - \angle OFp' = \tfrac{1}{2} \angle OFP' - \tfrac{1}{2} \angle OFP ;$$
$$\therefore \angle pFp' = \tfrac{1}{2} \angle PFP' \text{ and } \therefore \text{ constant.}$$

Cor.—Hence the anharmonic ratio of the pencil formed by joining four fixed points on a parabola to any fifth variable point is constant.

For $O . PP'P''P''' = O . pp'p''p''' = F . pp'p''p'''.$

PROPOSITION XIII.

The line joining the focus to the point of intersection of any two tangents to a parabola bisects the angle between the radii vectores drawn from the focus to the points of contact of the tangents.
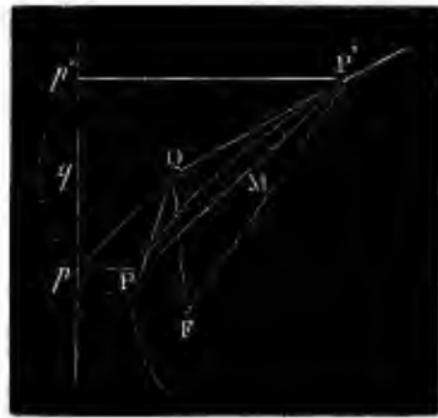

Fig. 18.

Let QP, QP' be tangents. Join $PF, P'F, QF.$
Draw $P'p', Pp \perp$ the directrix. Join $Qp, Qp'.$

Then $\triangle^s$ QFP' and $Qp'P'$ are equal; (4 I. Euclid.)

$\therefore QF = Qp'$, also $\angle\ QFP' = \angle\ Qp'P'$.

Again, $\triangle^s$ QFP and QpP are equal; (4 I. Euclid.)

$\therefore QF = Qp$, and $\angle\ QFP = \angle\ QpP$.

Hence, $Qp = Qp'$, $\therefore \angle Qpp' = \angle Qp'p$, $\therefore \angle QpP = \angle Qp P'$;

$$\therefore \angle QFP = \angle\ QFP'.$$

Cor. 1.—If Qq be drawn $\parallel$ the axis and produced it will bisect the chord PP'.

For $\triangle^s$ Qpq and $Qp'q$ are equal. (26 I. Euclid.)

$\therefore pq = qp'$; but Pp, Qq, and $P'p'$ are parallel;

$$\therefore PM = MP'.$$

Cor. 2.—Since $QF = Qp = Qp'$; $\therefore$ the circle described will centre Q, and radius QF will pass through p and p'.

Cor. 3.—By the aid of *Cor.* 2 we can draw a pair of tangents to a parabola from a point Q without it.

For, with centre Q, and radius QF, describe a circle from the points where this circle cuts the directrix; draw perpendiculars meeting the curve in P and P'; the lines joining Q with P and P' will be the required tangents.

Cor. 4.—$\triangle^s$ $P'QF$ and QPF are similar, and QF is a mean proportional between FP and FP'.

For $\angle\ FQP' = \angle\ P'Qp'$, $\angle\ FQP = \angle\ PQp$,

and $\angle\ pQq = \angle\ p'Qq$;

$$\therefore \angle\ FQP' + PQq = 180° ;$$

but $\angle pPQ$ or $\angle FPQ + \angle PQq = 180°$; (29 I. Euclid.)

$$\therefore \angle\ FQP' = \angle\ FPQ,$$

also $\angle\ QFP = \angle\ QFP'.$

Hence $\triangle\ P'QF$ is similar to $\triangle\ QPF$;

$$\therefore P'F : FQ :: FQ : FP.$$

PROPOSITION XIV.

The chord of contact of any two tangents to a parabola is an ordinate to the diameter passing through their point of intersection.
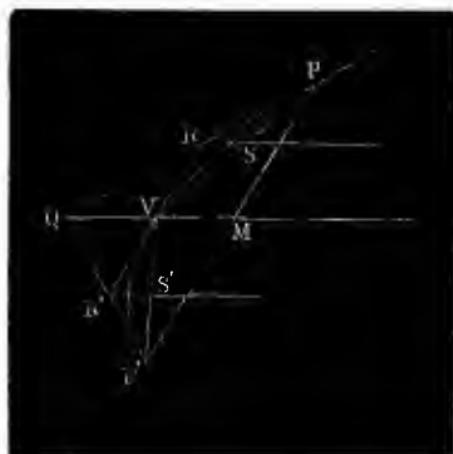
Fig. 19.

Let QP, QP', be two tangents, QM a parallel to the axis passing through their intersection, meeting the curve in V. At V draw a tangent meeting QP in R, and QP' in R'; join VP, VP'; through R and R' draw parallels to the axis.

Then VP will be bisected in S, and VP' in S'.

<div align="right">(Cor. 1, Prop. XIII.)</div>

Hence QP will be bisected at R, and QP' at R';

<div align="right">(2 VI. Euclid.)</div>

$$\therefore\ RR' \text{ is } \parallel PP'. \qquad \text{(2 VI. Euclid.)}$$

Hence PP' is an ordinate to diameter QM.   (*Def.*)

Cor. $QV = VM$.

Proposition XV.

The angle at the focus, subtended by the segment intercepted on a variable tangent to a parabola by two fixed tangents, is equal to half the angle which the chord of contact of the fixed tangents subtend at the focus, and therefore constant.
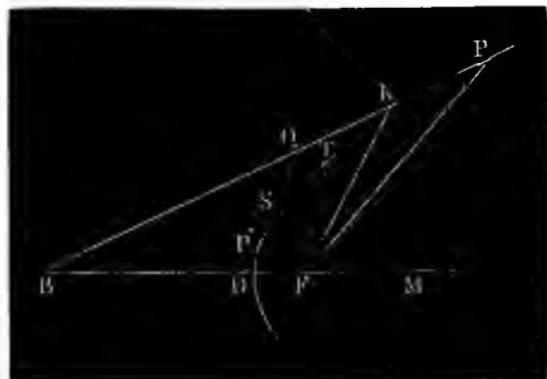
Fig. 20.

Let QP, QP', be the fixed tangents, RS the variable; join F with P', S, T, R, and P.

Then $\angle SFT = \frac{1}{2} \angle TFP'$; also $\angle TFR = \frac{1}{2} \angle TFP$;
(Prop. XIII.)

$\therefore \angle RFS = \frac{1}{2} \angle PFP'$, which is constant.

Cor.—The anharmonic ratio of the four points in which a variable tangent to a parabola is cut by four fixed tangents is constant.

For the segments intercepted on the variable tangents subtend constant angles at the focus.

Proposition XVI.

The circle circumscribing the triangle formed by any three tangents to a parabola passes through the focus.

For $\qquad \angle PFM = 2 \angle PBF,$ (*Cor.* 2, Prop. VIII.)

and $\qquad \angle P'FM = 2 \angle P'DF;$ (*Cor.* 2, Prop. VIII.)

$$\therefore \ \angle P'FM - \angle PFM = 2 \angle P'DF - 2 \angle PBF,$$

or $\qquad\qquad \angle P'FP = 2 \angle BQD;$

$$\therefore \ \angle SFR = \angle BQD. \qquad \text{(Prop. XV.)}$$

Hence a circle will pass through $SQRF$.

PROPOSITION XVII.

If a tangent at any point P on a parabola meet the axis in T and P, be joined with the vertex, the area of the triangle PAT thus formed will be equal to the area of the triangle PAp, formed by drawing Pp parallel to the axis, to meet the tangent at the vertex.
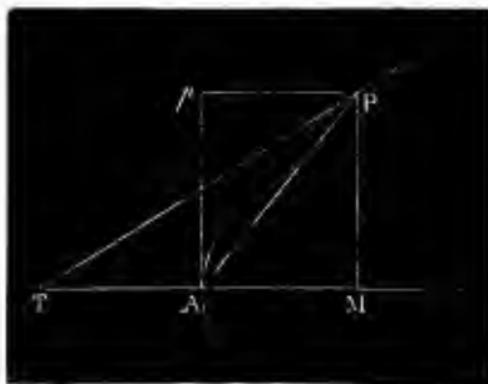

Fig. 21.

Draw the ordinate PM.

Then, since $\quad TA = AM;$ (Prop. VIII.)

$$\therefore \ \triangle TAP = \triangle MAP = \triangle PAp.$$

(38 I. Euclid.)

Cor. $\quad \triangle TMP =$ parallelogram $PMAp$.

PROPOSITION XVIII.

If QG be an ordinate drawn to any diameter Pp, from any point Q on a parabola cutting the axis in L, and QS be an ordinate drawn to the axis from the same point, AH a tangent at the vertex, then the triangle QLS = parallelogram $KSAH$.
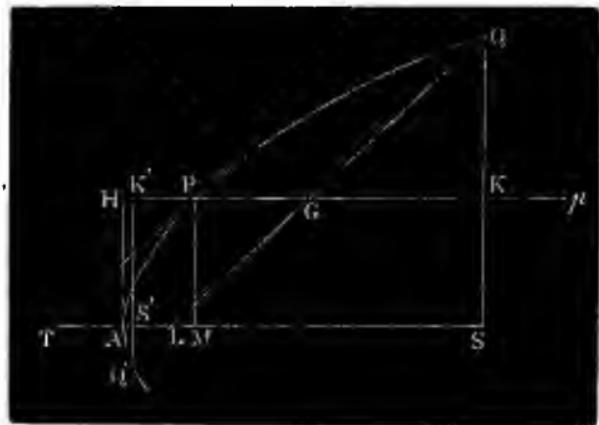
Fig. 22.

Draw PT the tangent at P.

Then $\triangle^s QLS$ and PTM are similar;

$$\therefore \ \triangle QLS : \triangle PTM = QS^2 : PM^2 \quad (19 \text{ VI. Euclid.})$$
$$= AS : AM \quad (Cor.\ 1, \text{Prop. IX.})$$
$$= \square ASKH : \square AMPH;$$

but $\qquad \triangle PTM = \square AMPH; \quad (Cor. \text{ Prop. XVII.})$

$$\therefore \ \triangle QLS = \square KSAH.$$

Cor. 1.—In like manner it may be proved that

$$\triangle Q'S'L = \square AS'K'H.$$

Cor. 2.—The chord QQ' is bisected by the diameter Pp.

Since $\triangle QLS = \square KSAH$; $\therefore$ $\triangle QGK = $ trap. $HALG$;

also $\triangle Q'LS' = \square K'S'AH$; $\therefore$ $\triangle Q'GK' = $ trap. $HALG$;

$\therefore$ $\triangle QGK = \triangle Q'GK'$; they are also similar;

$$\therefore \quad QG = Q'G.$$

Cor. 3. $\triangle QGK = \square GLTP$.

For $\triangle TPM = \square AMPH$;   (*Cor.* Prop. xvii.)

$\therefore$ $\square TPGL = $ trap. $HALG = \triangle QGK$.   (*Cor.* 2.)

Cor. 4.—Chords drawn parallel to the tangent at any point will be bisected by the diameter passing through that point.

This follows immediately from *Cor.* 2.

Proposition XIX.

If a line be drawn parallel to the chord of contact of two tangents, the parts intercepted on it between the curve and the tangents are equal.

Fig. 23.

Draw the diameter through Q.

Then PP' is an ordinate to QM;   (Prop. xiv.)

∴ *SS′*, which is ‖ *PP′*, is also an ordinate ;

∴ *PP* and *SS′* are bisected by *QMN*;
(*Cor.* 4, Prop. xviii.)
∴ *NT* = *NT′*.

Hence $TS = T'S'.$

Proposition XX.

The squares of the ordinates to any diameter are proportional to the abscissas.
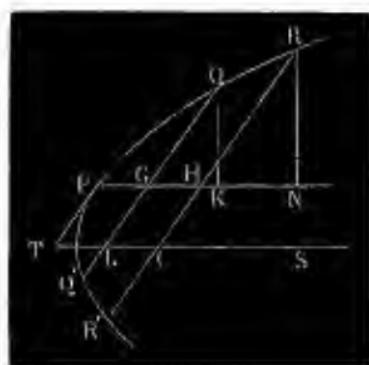
Fig. 24.

Let *TS* be the axis, *PN* any diameter, *QG*, *RH* ordinates ‖ the tangent at *P*.

Then △ *QGK* = ▢ *PGLT*,
(*Cor.* 3, Prop. xviii.)

and △ *RHN* = ▢ *PHCT*;

∴ △ *QGK* : △ *RHN* = ▢ *PGLT* : ▢ *PHCT*.

Hence $QG^2 : RH^2 = PG : PH.$
(19 & 1 VI. Euclid.)

PROPOSITION XXI.

The parameter of any diameter is four times the line joining the focus with the point where that diameter meets the curve.
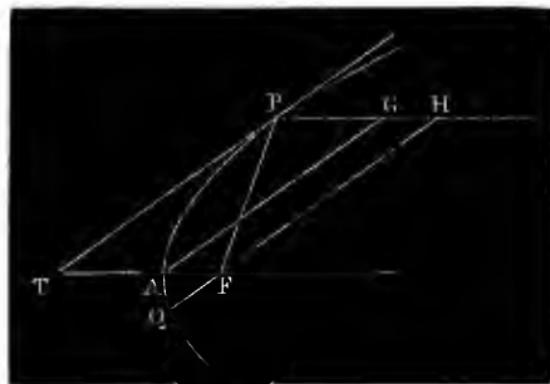
Fig. 25.

Let QH be the focal ordinate to the diameter PH. Draw $AG \parallel QH$.

Then $QH^2 : AG^2 = PH : PG = TF : TA$;
$$= 4TF \times TF : 4TF \times TA ;$$

but $\qquad AG^2 = PT^2 = 4TF \times TA$;

(*Cor.* 2, Prop. IX.)

$$\therefore QH^2 = 4TF \times TF;$$
$$\therefore QH = 2TF$$
$$= 2FP. \qquad \text{(Prop. VIII.)}$$

Hence,   parameter $= 4FP$.   (*Cor.* 4, Prop. XVIII.)

Cor.—The abscissa intercepted on any diameter by a double focal ordinate $= \frac{1}{4}$ of the parameter of that diameter.

For $\qquad PH = TF = FP = \frac{1}{4}$ parameter.

PROPOSITION XXII.

The square of the ordinate to any diameter is equal to the rectangle under the parameter of that diameter and the abscissa.

Fig. 26.

Let RS be any ordinate to the diameter PS, and QH the focal ordinate.

Then $\qquad RS^2 : QH^2 = PS : PH$; $\qquad$ (Prop. xx.)

but $\qquad\qquad QH = 2FP$ $\qquad\qquad$ (Prop. xxi.)

$\qquad\qquad\qquad = 2FT$ $\qquad\qquad$ (Prop. viii.)

$\qquad\qquad\qquad = 2PH$;

$\therefore RS^2 : 4PH^2 = PS : PH$

$\qquad\qquad\quad = 4PH \times PS : 4PH \times PH$;

$\qquad \therefore RS^2 = 4PH \times PS$

$\qquad\qquad\quad = 4FT \times PS$

$\qquad\qquad\quad = 4FP \times PS$

$\qquad\qquad\quad = \text{Parameter} \times \text{Abscissa}.$

$\qquad\qquad\qquad\qquad\qquad$ (Prop. xxi.)

PROPOSITION XXIII.

If from any point on a parabola a perpendicular be let fall on any diameter, the square of this perpendicular will be equal to the rectangle under the Latus rectum and the abscissa.

Fig. 27.

Draw AC the tangent at the vertex; join CF.

Then $\triangle^s\ QHG$ and CAT are similar;

$\therefore\ QH^2 : QG^2 = CA^2 : CT^2$;

$\therefore\ QH^2 : 4FP \times PG = FA \times AT : FT \times TA$

(Props. XXII. and IV.)

$= FA : FT$ or FP

$= 4FA \times PG : 4FP \times PG$;

$\therefore\ QH^2 = 4FA \times PG = $ Latus rectum $\times$ Abscissa.

NOTE.—The parameter of any diameter is proportional to the square of the cosecant of the angle which its ordinates make with the axis,

$$\text{cosec}^2 \angle\ QGH = \frac{QG^2}{QH^2} = \frac{4FP \times PG}{4FA \times PG} = \frac{4FP}{4FA};$$

$\therefore$ Latus rectum $\times$ cosec$^2 \angle\ QGH = 4FP = $ parameter of diameter PH.

PROPOSITION XXIV.

If a tangent be drawn at any point P on a parabola, and also an ordinate PM from same point to any diameter $Q'M$, the portion intercepted on that diameter between the tangent and the ordinated will be bisected by the curve.
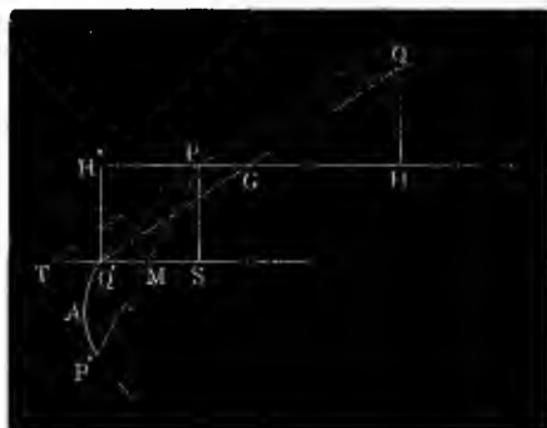
Fig. 28.

Let QQ' be the chord ∥ the tangent at P.

Draw QH ⊥ the diameter passing through P,

and PS ⊥ $Q'M$ produced.

Then $GQ = GQ'$;   (*Cor.* 4, Prop. XVIII.)

$$\therefore QH = Q'H' = PS;$$

but $QH^2 = $ Latus rectum $\times PG$,   (Prop. XXIII.)

and $PS^2 = $ Latus rectum $\times Q'M$;

$$\therefore Q'M = PG = TQ'.$$

Cor.—The tangents at the extremities of a double ordinate intersect on the diameter.

PROPOSITION XXV.

If any diameter intersect two parallel chords, the rectangles under the segments of these chords are proportional to the segments of the diameter intercepted between the chords and the curve.
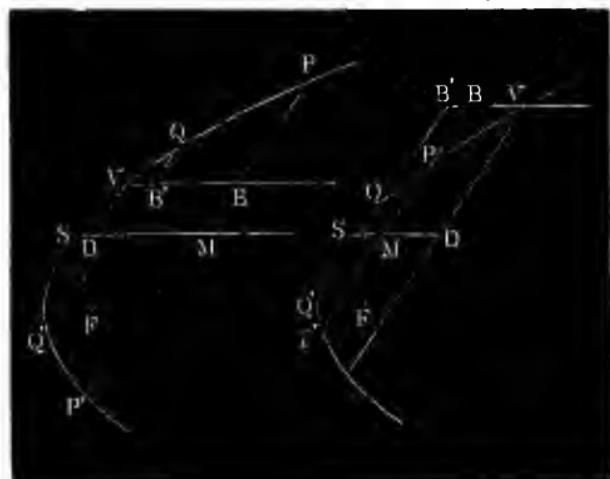
Fig. 29.   Fig. 30.

Draw the diameter SM, bisecting the parallel chords PP', QQ'.

Then $PM^2 = 4SF \times SM$,   (Prop. xxii.)

and $VD^2 = 4SF \times SD$;

$$\therefore PM^2 \sim VD^2 = 4SF \times DM,$$

or $PB \times BP' = 4SF \times VB$.

Similarly $QB' \times B'Q' = 4SF \times VB'$;

$$\therefore PB \times BP' : QB' \times B'Q' = VB : VB'.$$

Cor.—If a chord be drawn through any point, the rectangle under the segments is equal to the rectangle under the parameter of the diameter bisecting the chord, and the part intercepted on a diameter passing through the point between the curve and the point.

PROPOSITION XXVI.

If two chords of a parabola intersect, the rectangles under the segments are proportional to the parameters of the diameters bisecting the chords.
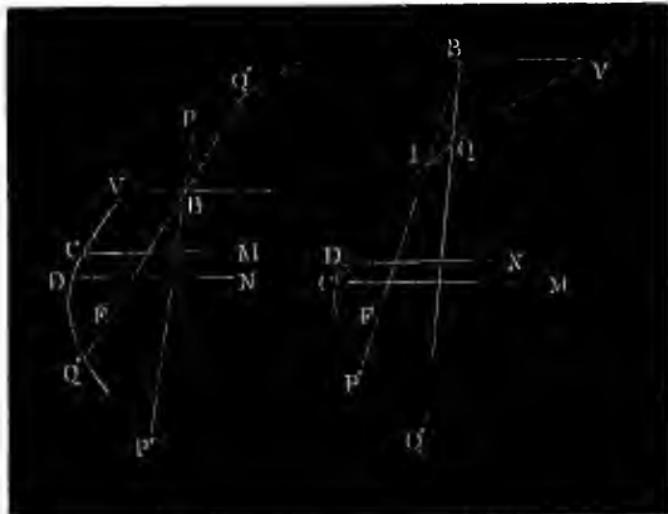
Fig. 31.   Fig. 32.

Let CM, DN, be the diameters bisecting the chords QQ', PP'. Draw the diameter BV through the intersection of the chords.

Then $\quad PB \times BP' = 4FD \times VB,\quad$ (*Cor.*, Prop. xxv.)

and $\quad\quad QB \times BQ' = 4FC \times VB;\quad$ (*Cor.*, Prop. xxv.)

$\quad\therefore\ PB \times BP' : QB \times BQ' :: 4FD : 4FC.$

Cor. 1.—The rectangles under the segments of any two intersecting chords are proportional to the lengths of the parallel focal chords.

Cor. 2.—If two intersecting chords be parallel to two others, the rectangles under the segments of the one pair are proportional to the rectangles under the segments of the other pair.

Cor. 3.—If the points *P* and *P′* coincide, the line *BP* becomes a tangent. Hence, if from any point two tangents be drawn to a parabola, and from any other point two parallel chords, the rectangles under the segments of the chords are proportional to the squares of the tangents.

Proposition XXVII.

If *TP* and *TQ* be two tangents to a parabola, *OC* a line drawn parallel to either, cutting the other in *O*, the curve in *A* and *C*, and the chord of contact of the tangents in *B*, then *OA*, *OB*, and *OC* are in geometric proportion.

Fig. 33.

For

$$OA \times OC : OQ^2 = TP^2 : TQ^2$$

$$= OB^2 : OQ^2;$$

(Cor. 3, Prop. xxvi.)

$$\therefore OA \times OC = OB^2;$$

$$\therefore OA : OB = OB : OC.$$

D

PROPOSITION XXVIII.

If a circle intersect a parabola in four points, the common chords will be equally inclined to the axis.

Fig. 34.

Let PP', QQ' be the common chords intersecting in O.

Then $OP \times OP' : OQ \times OQ' =$ the ratio of the parallel focal chords,   (*Cor.* 1, Prop. XXVI.)

but $OP \times OP' = OQ \times OQ'$;

                                (35 III. Euclid.)

∴ the focal chords drawn parallel to QQ' and PP are equal.

Hence the focal chords, and ∴ the chords QQ' and PP' are equally inclined to the axis.   (*Cor.* 2, Prop. I.)

Cor. 1.—In like manner it can be shown that PQ and $P'Q'$, also PQ' and $P'Q$ are equally inclined to the axis.

PROPOSITION XXIX.

Any line *OB* drawn through the intersection of two tangents to a parabola is cut harmonically by the curve, the point, and the chord of contact of the tangents.

Fig. 35.

Through *B* and *B'* draw *TS* and *T'S'* ∥ *PQ*, the chord of contact of the tangents.

Then $BO : OB' = BT : B'T' = BS : B'S'$;
$$\text{(Similar } \triangle^s.)$$
$$\therefore BO^2 : OB'^2 = BT \times BS : B'T' : B'S',$$
$$= BT \times TA : B'T' : T'A',$$
$$\text{(Prop. xix.).}$$
$$= TP^2 : PT'^2,$$
$$= BO^2 : O'B^2;$$
$$\therefore BO : OB' = BO' : O'B'.$$

Cor.—If through any point O without a parabola a line be drawn cutting the curve in B and B'; and OO' be taken equal to the harmonic mean between OB and OB', the locus of O' is a straight line, namely, the chord of contact of the tangents drawn from O.

Proposition XXX.

If PM, QN be any diameters, BE a line drawn parallel to the ordinates to either, meeting the curve in D and QP in C, then BC, BD, and BE are in geometric proportion.

Fig. 36.

Draw the ordinate QG.

Then $\quad BE^2 : BD^2 = GQ^2 : BD^2$,

$$= PG : PB, \qquad \text{(Prop. xx.)}$$
$$= PQ : PC, \qquad \text{(2 VI. Euclid.)}$$
$$= BE : BC,$$
$$= BE^2 : BE \times BC;$$
$$\therefore BD^2 = BE \times BC.$$

PROPOSITION XXXI.

If through any point P on a parabola lines POO', $Q'PQ$, be drawn parallel to any two adjacent sides DA, DC, of an inscribed quadrilateral, meeting opposite sides in O, O', and Q, Q', then $PO \times PO' : PQ \times PQ'$ in a constant ratio.
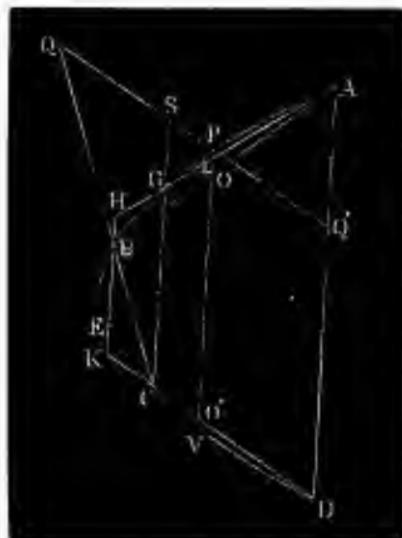
Fig. 37.

Through B and C draw BE and $CG \parallel$ to AD; join AG, and produce it to meet EB produced in H.

$$OL : BH = LA : AH$$
$$= O'D : DK. \quad \text{(Parallel lines.)}$$

Alternation $\quad OL : O'D = BH : DK.$

Also $\quad SC$ or $PO' : SQ = BK : KC.$ $\quad$ (Similar $\triangle^s$.)

$\therefore \; OL \times PO' : O'D \times SQ = BH \times BK : DK \times KC;$

$\therefore \; OL \times PO' : PQ' \times SQ = EK \times BK : DK \times KC^*$
$$= O'V \times O'P : O'C \times O'D$$
$$= PL \times O'P : PS \times PQ';^*$$

---

* The diameter which bisects the parallel chords GC and AD will bisect the chords BE and PV (*Cor.* 4, Prop. xviii.), also the parallel lines HK and LO'. Hence $BH = EK$, and $O'V = PL$.

$$\therefore\ OL \times PO' + PL \times O'P : PQ' \times SQ + PS \times PQ'$$
$$= EK \times BK : DK \times KC;$$

$$\therefore\ PO \times PO' : PQ \times PQ' = EK \times BK : DK \times KC.$$

Now it is evident that the points A, B, C, D being fixed, E is also fixed, and $\therefore\ EK \times BK : DK \times KC$ in a constant ratio, and $\therefore\ PO \times PO' : PQ \times PQ'$ in a constant ratio.

Proposition XXXII.

If from any point on a parabola lines PR, PR', PS, PS' be drawn to the sides of an inscribed quadrilateral, making with them any constant angles, then the rectangles under the lines drawn to the opposite sides will be in a constant ratio.
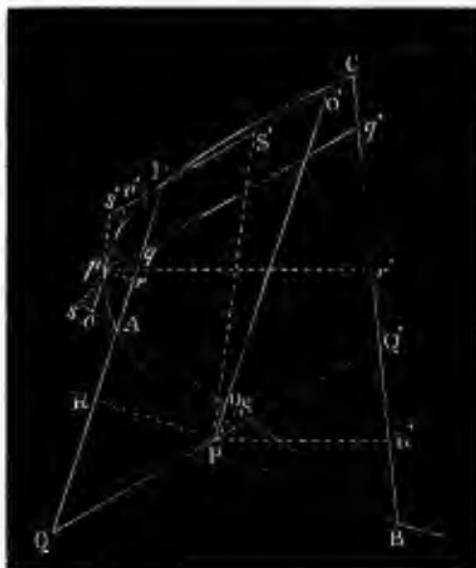

Fig. 38.

Take any other point p on the curve.

Through the points P, p, draw QPQ' and $pqq' \parallel DC$ and POO', $opo' \parallel AD$, also pr, pr', ps, ps', $\parallel PR$, PR', PS, PS', respectively.

Then $\qquad PR : pr = PQ : pq, \qquad$ (Similar $\triangle^s$).

and $\qquad\qquad PR' : pr' = PQ' : pq'$;

$\qquad \therefore PR \times PR' : pr \times pr' = PQ \times PQ' : pq \times pq'.$

Similarly it may be proved that

$\qquad PS \times PS' : ps \times ps' = PO \times PO' : po \times po'$;

but $\qquad PQ \times PQ' : PO \times PO' = pq \times pq' : po \times po'$;

$\qquad \therefore PR \times PR' : PS \times PS' = pr \times pr' : ps \times ps'.$

Cor. 1.—The rectangle under the perpendiculars let fall from any point of a parabola, on two opposite sides of an inscribed quadrilateral, is a constant ratio to the rectangle under the perpendiculars let fall on the other two sides.

Cor. 2.—If the points A and B coincide, also the points C and D, then the sides AB and CD become tangents, and the sides BC and AD coincide and become the chord of contact. Then the rectangle under the perpendiculars let fall from any point on a parabola on two fixed tangents is in a constant ratio to the square of the perpendicular let fall on their chord of contact.

Def.—Two curves are said to be *similar* and *similarly situated*, when a radius vector drawn from a fixed point in any direction to the first curve bears a constant ratio to the radius vector drawn from a fixed point in a parallel direction to the second curve.

Def.—Two curves are *similar*, when a radius vector drawn from a fixed point to the first curve bears a constant ratio to the radius vector drawn from a fixed point to the second curve, in a direction inclined at a constant angle to the former.

The two fixed points are called *centres of similarity*.

PROPOSITION XXXIII.

All parabolas are similar figures.

Fig. 39.

Let F, f, be the foci, A, a, the vertices. Draw any two lines FP, fp, making equal $\angle^s$ with the axes. Draw tangents to the curves at P and p; let fall perpendiculars FT, ft, on the tangents from the foci; join TA, ta.

Then TA, ta, will be tangents at the vertices.

(Prop. IV.)

Hence also $\angle PFT = \frac{1}{2} \angle PFA$, and $\angle pft = \frac{1}{2} \angle pfa$;

(Prop. XIII.)

$$\therefore \angle PFT = \angle pft.$$

Hence $\triangle PFT$ is similar to pft.

Also $\triangle TFA$ is similar to $\triangle tfa$;

$\therefore PF : FT = pf : ft$, and $TF : FA = tf : fa$;

$\therefore PF : FA : : pf : fa.$

Alt. $PF : pf : : FA : fa$;

$\therefore$ The curves are similar.

Proposition XXXIV.

The area included between any ordinate *PM*, the abscissa *AM*, and the curve, is two-thirds of the parallelogram which has the ordinate and abscissa for its adjacent sides.

Fig. 40.

Draw tangents at *A* and *P*; join *AP*.

Since $TA = AM$, and $\therefore$ $TH = HP$,   (Prop. xxiv.)

$$\therefore \quad \triangle TAH = \tfrac{1}{2} \triangle TAP = \tfrac{1}{2} \triangle APM.$$

Similarly, if through *H* we draw a diameter, and at the point *Q* where it meets the curve draw a tangent, we can prove that $\triangle HQh = \tfrac{1}{2} \triangle QGP$, and $\triangle HQh' = \tfrac{1}{2} \triangle QGA$. Continuing in this way to form new $\triangle^s$ by drawing diameters through *h* and *h'*, we can prove that the areas of the exterior $\triangle^s$ formed by the tangents are the halves of the areas of the interior $\triangle^s$, formed by joining the points of contact with the extremities of the chords.   The same will hold good though the number of $\triangle^s$ be increased indefinitely;

∴ the sum of all the exterior Δ^s = half the sum of the interior Δ^s.

Hence the area included between the curve and the lines AM and AP = $\frac{2}{3}$ area of $\triangle$ TPM = $\frac{2}{3}$ parallelogram $AMPB$.

Otherwise thus—
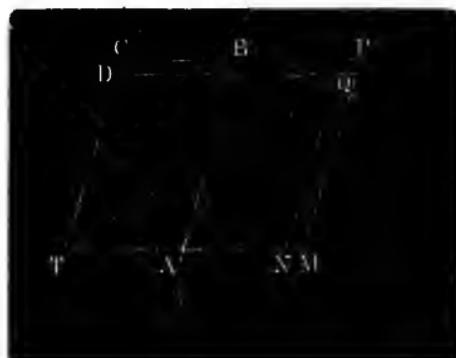
Fig. 41.

Draw the adjacent ordinate QN; join PQ and produce it; draw the tangent at A; draw $TC \parallel AB$, and $QD \parallel TM$.

Then, since PM is $\parallel AB$, the $\square$ $QM = QC$.

Now when Q is indefinitely near P, PT becomes the tangent at P, and (Prop. XIV., *Cor.*) $AM = AT$.

$$\therefore \ \square \ BQ = \square \ BD \ ;$$

$$\therefore \ \square \ QM = 2 \ \square \ BQ.$$

If the parabolic arc AP be divided into an indefinite number of small parts, it can be similarly shown that each internal parallelogram is double the external, and hence the total internal area APM is double of the external area APB, and ∴ = $\frac{2}{3}$ of $\square$ $AMPB$.

Def.—If a right-angled triangle be made to revolve round one of the sides containing the right angle, the surface generated by the hypothenuse is called a *right cone*, and the hypothenuse the *generating line*.

Proposition XXXV.

The *section* of a right cone by a plane parallel to the *generating line* is a parabola.
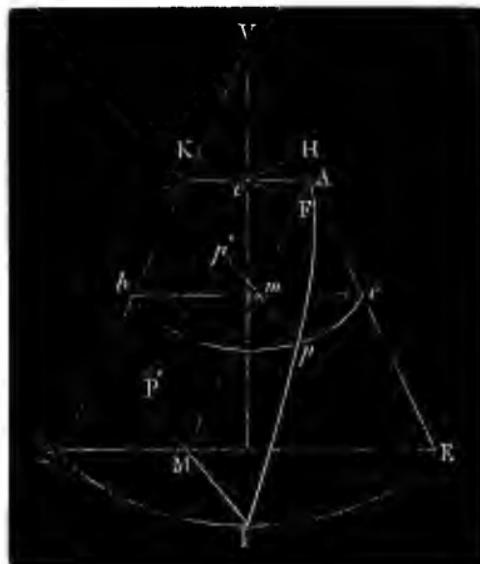
Fig. 42.

Let the plane BVE, drawn through the axis of the cone, perpendicular to the plane of the *section*, coincide with the plane of the paper, then both the *section* $P'AP$ and the base BPE will be perpendicular to the plane of the paper; therefore the line MP in which the *section* cuts the base is perpendicular to the plane of the paper, and, therefore, perpendicular to BE, the diameter of the base.

Hence $BM \times ME = MP^2$.   (35 III. Euclid.)

If now any other plane *bpe* be drawn parallel to the base, meeting the *section* in *pm*, it can similarly be shown that *mp* is perpendicular to *be*, and $\therefore$ $bm \times me = mp^2$;

$$\therefore MP^2 : mp^2 = BM \times ME : bm \times me$$

$$= ME : me$$

$$= MA : mA. \qquad \text{(Similar } \Delta^s.)$$

Hence the section $P'AP$ is a parabola.

Cor. 1.—The latus rectum is a third proportional to VD and DA.

For $VD : DA = AM : ME$

$$= AM \times MB : ME \times MB$$

$$= AM \times MB : MP^2$$

$$= AM \times DA : \text{Latus rectum} \times AM ;$$

$$\text{(Prop. ix.)}$$

$$\therefore VD : DA = DA : \text{Latus rectum.}$$

Cor. 2.—If from O a perpendicular be let fall on the section, the foot of this perpendicular will be the focus.

For Δ^s VDO and OAF are equiangular ;

$$\therefore VD : 2DO = 2OA : 4AF;$$

$$\therefore VD : DA = DA : 4AF;$$

$$\therefore 4AF = 4 \text{ Latus rectum.} \qquad (Cor. 1.)$$

Hence F is the focus.   (*Cor.* 2, Prop. i.)

PROBLEMS ON THE PARABOLA.

1. If from any point on a tangent to a parabola, a perpendicular be let fall on the radius vector drawn from the focus to the point of contact of the tangent, the segment intercepted on the radius vector, between the foot of the perpendicular and the focus, is equal to the perpendicular let fall from the point on the directrix.

2. The focus and a tangent being given, the locus of the vertex is a circle.

3. Find the locus of the point of intersection of any tangent to a parabola, with the line drawn from the focus, making a constant angle with the tangent.

If the vertex of a triangle of given species be fixed, while one base angle moves along a fixed right line, the locus of the other base angle will be a right line. (See also *Cor.* 1, Prop. IV.)

4. Given four tangents to a parabola, determine the focus and directrix. (See Prop. XVI.)

5. Normals at the extremities of a focal chord intersect on the diameter which bisects the chord. (See Prop. X.)

6. The circle described on any focal chord as diameter touches the directrix.

7. If a triangle ABC circumscribe a parabola whose focus is F, the lines drawn through A, B, C, perpendicular to FA, FB, and FC respectively, will pass through a point.

8. If a circle intersect a parabola in four points, the sums of the ordinates of the points of intersection on opposite sides of the axis are equal to each other.

9. Show how to cut from a cone a parabola having a given Latus rectum.

10. Prove that the lines joining the middle points of the sides of a triangle self-conjugate to a parabola are tangents to the curve. (See Props. XXIX. and XXIV.)

CHAPTER II.

THE ELLIPSE.

DEFINITIONS.

AN Ellipse is the curve traced out by a point, which moves in such a way that its distance from a fixed point is to its perpendicular distance from a fixed right line in a constant ratio $\varepsilon : 1$ (ε being less than unity).

The fixed point is called the *Focus*, and the fixed right line the *Directrix*.

The right line drawn through the Focus perpendicular to the directrix is called the *Axis*.

The points at which the axis meets the curve are called the *Vertices*.

The middle point of the portion of the axis intercepted between the vertices is called the *Centre*.

Any line drawn through the centre is called a *Diameter*.

A right line which meets the curve, and being produced, does not cut it, is called a *Tangent*.

A right line drawn through any point on the curve perpendicular to the tangent at that point is called a *Normal*.

The right line joining any two points on the curve is called a *Chord*.

For Definitions of *Ordinate* and *Abscissa* see Definitions, Chapter I.

LEMMA.

If a right line AB be divided internally at O in any ratio, and externally at O' in the same ratio, and a circle be described on OO' as diameter, the right lines joining any point P on this circle with the extremities of the line AB will have the same ratio.

Fig. 1.

Bisect $\qquad OO'$ in C; join CP, PO.

Then $\qquad AO' : O'B = AO : OB$;

$\qquad AO' + AO : AO' - AO = O'B + OB : O'B - OB$;

$\therefore\ 2AC : 2OC = 2OC : 2BC$;

$\therefore\ AC : CP = CP : CB$;

$\therefore\ \triangle ACP$ is similar to $\triangle PCB$; $\qquad$ (6 VI. Euclid.)

$\therefore\ \angle CPB = \angle CAP$;

but $\qquad \angle CPO = \angle COP$ $\qquad$ (5 I. Euclid.)

$\qquad\qquad = \angle OAP + \angle OPA$ $\qquad$ (32 I. Euclid.)

$\qquad\qquad = \angle CPB + \angle OPA$;

$\therefore\ \angle BPO = \angle OPA$;

$\therefore\ AP : PB = AO : OB$. $\qquad$ (3 VI. Euclid.)

Proposition I.

The focus, directrix, and eccentricity of an ellipse being given, to determine any number of points on the curve.
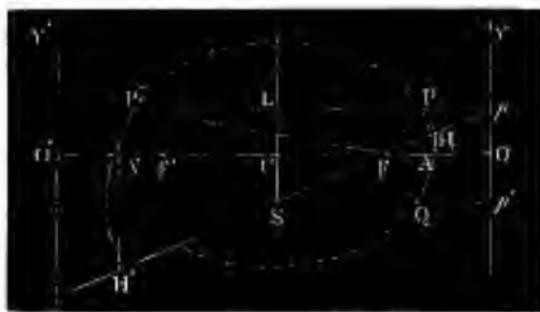
Fig. 2.

Let F be the focus, and Oy the directrix. Draw $FO \perp$ the directrix; divide FO internally at A, and externally at A', so that $FA : AO :: FA' : A'O$ in the given ratio, $\varepsilon : 1$; then A and A' are the vertices of the curve, and AA' the axis.   (*Def.*)

On the directrix take any point p; join Fp; draw AH and $A'H' \perp AA'$, meeting pF in H and H'; on HH' as diameter describe a circle; through p draw $pPP' \parallel$ the axis cutting the circle in P, P'; join PF, $P'F$.

Then since AH is $\parallel Op$; $\therefore HF : Hp :: AF : AO$;

also since $A'H'$ is $\parallel Op$; $\therefore H'F : H'p :: A'F : A'O$;

$$\therefore H'F : H'p :: HF : Hp.$$

Hence $PF : Pp = FH : Hp$ (Lemma.)

                     $= FA : AO = \varepsilon : 1$.

                             (2 VI. Euclid.)

Also $P'F : P'p = FH : Hp$ (Lemma.)

                     $= FA : AO = \varepsilon : 1$;

                             (2 VI. Euclid.)

$\therefore P$ and P' are points on the curve.

In like manner, by taking other points on the directrix, any number of points on the curve may be determined.

Cor. 1.—If Op' be taken equal to Op, and another point Q found in a similar manner to P, it is obvious that Qp' will be equal to Pp; therefore corresponding to any point P, at a perpendicular distance from the axis $= Op$, there is another point Q on the other side of the axis. at a distance from it $= Op$, and also at the same distance as P from the directrix; hence the curve is symmetrical with regard to the axis.

Cor. 2.—Bisect AA' in C; through C draw $LCS \perp AA'$ and PP', meeting HH' in S, and PP' in L.

Then since AH, CS, and $A'H'$ are parallel;

$$\therefore SH = SH'.$$

Hence S is the centre of the circle $H'P'PH$;

and $\qquad\qquad \therefore LP = LP'.$ (3 III. Euclid.)

Therefore corresponding to any point P on the curve there is another point P' on the other side CL, situated in precisely the same manner with regard to it as P; hence the curve is symmetrical also with regard to CL.

If, therefore, CO' be measured off $= CO$, and $CF' = CF$, and $O'y'$ be drawn $\perp O'C$, the curve could be equally well described with F' as focus, and $O'y'$ directrix.

Cor. 3. $PF : Pp = AF : AO = A'F : A'O$;

$$\therefore PF : Pp = AF + A'F : AO + A'O$$
$$= 2CA : 2CO;$$
$$\therefore PF : Pp = CA : CO,$$

that is, the distance of any point on the curve from the focus is to its distance from the directrix as $CA : CO$.

Cor. 4. $A'F : A'O = AF : AO$;

$$\therefore A'F + AF : A'F - AF = A'O + AO : A'O - AO;$$
$$2CA : 2CF = 2CO : 2CA;$$
$$\therefore CF \times CO = CA^2.$$

E

Cor. 5.—From the symmetry of the curve it is evident that any chord drawn through C is bisected at that point. From this property the point C is called the *centre* of the curve.

Def.—The chord drawn through the centre at right-angles to the Axis, or *Axis major*, as that portion of the axis intercepted between the vertices is frequently termed, is called the *Axis minor*.

PROPOSITION II.

The sum of the distances of any point P on an ellipse from the foci is constant and equal to the axis-major.
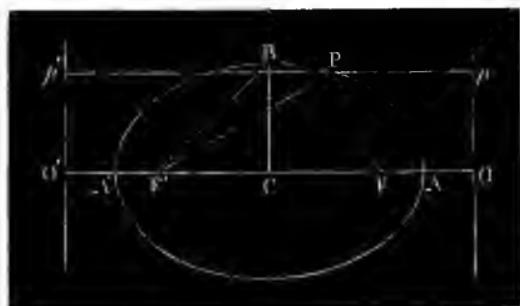
Fig. 3.

$$PF : Pp = CA : CO.$$

(*Cor.* 3, Prop. I.)

Also $$PF' : Pp' = CA : CO ;$$

$$\therefore PF + PF' : Pp + Pp' = CA : CO.$$

But $Pp + Pp' = 2CO$; $\therefore PF + PF' = 2CA = AA'$.

Cor. 1.—By the help of the preceding theorem we can describe an ellipse mechanically.

If the extremities of a thread, equal in length to AA', be fastened to the fixed points F, F', it is obvious that a pencil, moved about so as to keep the thread always stretched, will describe an ellipse whose foci are F and F'.

Cor. 2.—The distance of the extremity of the axis minor from either focus is equal to the semiaxis major. Let CB be the semiaxis minor; join BF, BF'.

Then the $\triangle^s BCF$ and BCF' are equal;

(4 I. Euclid.)

$$\therefore BF = BF' = \tfrac{1}{2}(BF + BF') = \tfrac{1}{2}AA' = CA.$$

Cor. 3.—$BC^2 = BF^2 - FC^2$

$$= CA^2 - CF^2 = A'F \times FA.$$

Cor. 4.—The sum of the distances of any point from the foci of an ellipse is greater or less than the axis major, according as the point is without or within the ellipse.
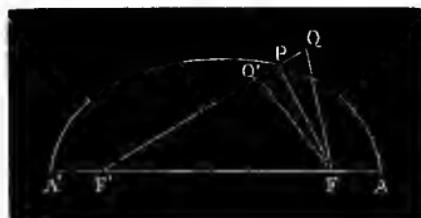
Fig. 4.

First let the point Q be without the ellipse; join QF, QF', and let QF' meet the ellipse in P; join PF.

Then $\qquad QF + QP > PF$;

$$\therefore QF + QF' > PF + PF'$$

$$> AA'.$$

Next, let Q' be within the ellipse; join $Q'F$, $Q'F'$, and let $F'Q'$ produced meet the ellipse in P; join PF.

Then $\qquad Q'F < Q'P + PF$;

$$\therefore Q'F + Q'F' < PF' + PF$$

$$< AA'.$$

E 2

Cor. 5.—Conversely, a point will be within or without an ellipse according as the sum of its distances from the foci is less than or greater than the axis major.

PROPOSITION III.

1°. The line which bisects the external angle formed by drawing lines from any point on an ellipse to the foci falls wholly without the curve.

2°. Any other line drawn through the point will cut the curve.

Fig. 5.

1°. Let PH bisect the angle between PF and PF' produced.

Let Q be any point on PH.

Cut off $PR = PF$.

Join QF, QF', QR.

Then the $\triangle^s QFP$ and QRP are equal ; (4 I. Euclid.)

$$\therefore \; QR = QF;$$

but $\qquad F'Q + QR > F'R;$ \qquad (20 I. Euclid.)

$$\therefore \; F'Q + QF > F'P + PF$$

$$> \text{axis major};$$ \qquad (Prop. ii.)

$\therefore \; Q$ is a point without the curve.

(*Cor.* 5, Prop. ii.)

2°. Let *PK* be any line other than the bisector of the angle *FPR*.

Draw *PL*, making the ∠ *KPL* = ∠ *KPF*; cut off *PL* = *PF*; join *F'L*, and where it cuts *PK* join with *F*.
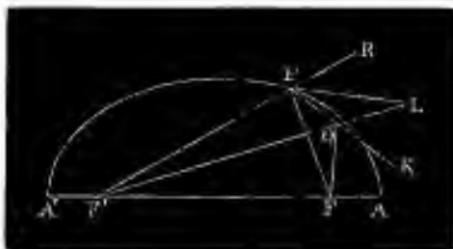
Fig. 6.

Then the $\triangle^s$ *FPQ* and *LPQ* are equal;   (4 I. Euclid.)

$$\therefore\ QF = QL;$$

but $F'P + PL > F'L$;   (20 I. Euclid)

$$\therefore\ F'P + PF > F'Q + QF;$$

$$\therefore\qquad AA' > F'Q + QF;$$

∴ the point *Q* is within the ellipse.   (*Cor.* 5, Prop. II.)

Therefore the line *PK* will cut the curve.

Cor. 1.—Hence the line which bisects the external angle between lines drawn from any point on an ellipse to the foci is a tangent to the curve.

Cor. 2.—The tangent at the vertex is perpendicular to the axis, and hence ordinates to the axis are also perpendicular to the axis.

Cor. 3.—The normal to an ellipse at any point bisects the angle between the focal radii vectores drawn to the point.

Proposition IV.

The locus of the foot of the perpendicular let fall from either focus on any tangent to an ellipse is a circle described on the axis major as diameter.

Fig. 7.

Let PH be any tangent, and FT a perpendicular let fall on it from the focus.

Produce $F'P$ to meet FT produced in R; join TC.

Then the $\triangle^s PFT$ and PRT are equal;

(26 I. Euclid.)

$$\therefore FT = TR.$$

Now in $\triangle F'FR$, $F'F$ is bisected in C, and FR in T.

$$\therefore CT \text{ is } \parallel F'R, \qquad \text{(2 VI. Euclid.)}$$

and
$$CT = \tfrac{1}{2} F'R \qquad \text{(Similar } \triangle^s\text{.)}$$
$$= \tfrac{1}{2} (F'P + FP)$$
$$= \tfrac{1}{2} AA'$$
$$= CA.$$

Hence the locus of T is a circle described with the centre C, and radius CA.

This circle is called *the Auxiliary circle* of the ellipse.

Cor. 1.—Conversely the right line drawn from either focus to the adjacent point of intersection of any tangent with the auxiliary circle is perpendicular to the tangent.

Cor. 2.—If the vertex of a right angle FTP move along a fixed circle, while one leg passes through a fixed point F within that circle, the other leg will always touch an ellipse.

Cor. 3.—By the aid of this Prop. we can draw a pair of tangents to an ellipse from any point Q without it.

On the line joining the given point Q, with either focus as diameter, describe a circle ; the lines joining Q with the points of intersection of this circle with the auxiliary circle will be the required tangents.

NOTE.—The circle described on QF as diameter will always cut the auxiliary circle.

Fig. 8.

For, bisect QF in O; join OC, QF'.

Then, since $\qquad QF + QF' > AA'$;   (*Cor.* 4, Prop. III.)

$$\therefore\ OF + OC > CA ;$$

$$\therefore\ OC > CA - OF.$$

That is, the distance between the centres of the circles is greater than the difference of their radii ; $\therefore$ they will intersect.

Cor. 4.—The right line drawn from the centre parallel to either focal radius vector, to meet the tangent, is equal to the semiaxis major.

Cor. 5.—The circle described on any focal radius vector as diameter will touch the auxiliary circle.

PROPOSITION V.

The rectangle under the focal perpendiculars on any tangent to an ellipse is equal to the square of the semi-axis minor.
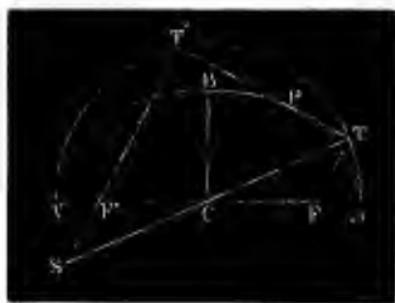
Fig. 9.

Let FT, $F'T'$, be perpendiculars let fall from the foci on the tangent at any point P.

Join TC, and produce it to meet $T'F'$ produced in S.

Then the $\triangle^s F'CS$ and FCT are equal; (26 I. Euclid.)

$\therefore F'S = FT$, and $CS = CT = CA$;   (Prop. IV.)

$\therefore S$ is a point on the auxiliary circle ;

$\therefore A'F' \times F'A = SF' \times F'T'$ (35 III Euclid.)

$$= FT \times F'T' ;$$

$\therefore A'C^2 - CF'^2 = FT \times F'T',$ (5 II. Euclid.)

or $BC^2 = FT \times F'T'.$ (*Cor.* 3, Prop. II.)

PROPOSITION VI.

The locus of the intersection of the tangents to an ellipse which cut at right angles is a circle.
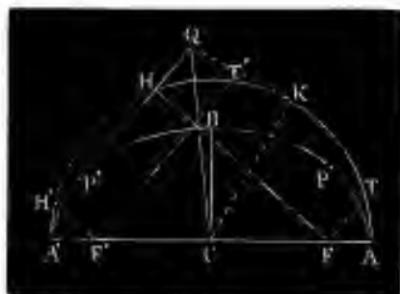
Fig. 10.

Let QP, QP', be tangents, which cut at right angles. Describe the auxiliary circle cutting the tangent QP in T and T', and the tangent QP' in H and H'; join FH, FT, $F'H'$, $F'T'$; draw QK touching the auxiliary circle; join CK, CQ.

Then FT and $F'T'$ are both $\perp QP$, and $\therefore \parallel QP'$.

(*Cor.* 1, Prop. IV.)

Similarly, $F'H'$ and FH are both $\perp QP'$, and $\therefore \parallel QP$;

(*Cor.* 1, Prop. IV.)

hence $\qquad QT = FH$, and $QT' = F'H'$.

Now $\qquad CQ^2 = CK^2 + QK^2$

$$= CA^2 + QT \times QT' \quad \text{(36 III. Euclid.)}$$

$$= CA^2 + FH \times F'H'$$

$$= CA^2 + CB^2 \qquad \text{(Prop. V.)}$$

$$= AB^2;$$

$\therefore \ CQ = AB.$

Hence the locus of Q is a circle described with the centre C, and radius $= AB$.

This circle is called *the Director Circle* of the ellipse.

PROPOSITION VII.

If PN be a normal to an ellipse at any point P, and PM an ordinate to the axis, then

$$CM : CN = CA^2 : CF^2.$$

Fig. 11.

Join PF, PF'; draw $p'Pp \parallel$ the axis major, meeting the directrices in p', p.

Since (*Cor.* 3, Prop. III.) PN bisects the $\angle F'PF$;

$$\therefore NF' : NF = PF' : PF \qquad (3 \text{ VI. Euclid.})$$
$$= Pp' : Pp.$$

Comp. and div.

$$NF' + NF : NF' - NF = Pp' + Pp : Pp' - Pp,$$

or
$$2CF : 2CN = 2CO : 2CM;$$
$$\therefore CM : CN = CO : CF$$
$$= CO \times CF : CF^2.$$

Hence
$$CM : CN = CA^2 : CF^2.$$
(*Cor.* 4, Prop. I.)

Cor. 1.—$CM : MN = CA^2 : CB^2$.

For $\qquad CM : CM - CN = CA^2 : CA^2 - CF^2,$

or $\qquad CM : MN = CA^2 : CB^2.$

Cor. 2. $F'P : PF = F'N : NF$

$$\therefore F'P + PF : PF = F'N + NF : NF,$$

or $CA : CF = PF : NF.$

Similarly, $CA : CF = PF' : NF';$

$$\therefore CA^2 : CF^2 : : PF \times PF' : NF \times NF'$$

$$\therefore CA^2 : CA^2 - CF^2 : : PF \times PF' : PF \times PF' - NF \times NF'$$

$$CA^2 : CB^2 : : PF \times PF' : PN^2.$$

PROPOSITION VIII.

If a tangent at any point P of an ellipse be produced to meet the axis in T, and the ordinate PM be drawn, then CA will be a mean proportional between CM and CT.

Fig. 12.

Through P draw $p'Pp \parallel$ the axis; join PF, PF'.
Then since PT is the bisector of the $\angle FPH$;

$$\therefore F'T : TF = F'P : PF \quad (Cor. 3, \text{VI. Euclid.})$$

$$= p'P : Pp.$$

Comp. and div.

$$F'T + TF : F'T - TF = Pp' + Pp : Pp' - Pp,$$

or $2CT : 2CF = 2CO : 2CM;$

$$\therefore CT \times CM = CF \times CO$$

$$= CA^2. \quad (Cor. 4, \text{Prop. I.})$$

Cor. 1.—If the auxiliary circle be described, and PM produced to meet it in Q, then QT will be a tangent to the circle.

For $CT : CA = CA : CM$;

$\therefore CT : CQ = CQ : CM$;

$\therefore$ the $\triangle^s CQT$ and CMQ are similar;
(6 VI. Euclid.)

$\therefore \angle CQT = \angle CMQ = 90°$;

$\therefore QT$ is a tangent to the circle.

Cor. 2.—Tangents to the ellipse and auxiliary circle, respectively, at the points where they are cut by any ordinate to the axis, will meet the axis produced in the same point.

Cor. 3.—Corresponding ordinates PM, QM, of the ellipse and auxiliary circle are in a constant ratio.

Since the $\triangle NPT$ is right-angled, and $PM \perp NT$;

$$\therefore NM \times MT = PM^2.$$

Also, since $\triangle CQT$ is right-angled, and $QM \perp CT$;

$$\therefore CM \times MT = QM^2.$$

Hence $PM^2 : QM^2 = NM : CM$

$$= CB^2 : CA^2.$$
(*Cor.* 1, Prop. vii.)

$$\therefore PM : QM = CB : CA.$$

Note.—The angle QCA is called *the excentric angle* of the point P.

Cor. 4.—The ellipse is concave towards the axis major.
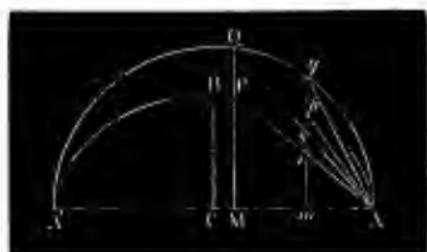
Fig. 13.

$$QM : PM = AC : BC = qm : pm. \quad (Cor. 3.)$$

But $\quad QM : PM = sm : rm; \quad$ (Similar $\triangle^s$.)

$$\therefore qm : pm = sm : rm.$$

But qm is $> sm$ by property of the circle;

$$\therefore pm \text{ is} > rm.$$

That is, if a perpendicular be drawn to the axis major, between the vertex A and any point P on the curve, the segment intercepted on it between the curve and the axis major is greater than the segment intercepted on it between the line AP and the axis major; and, therefore, the curve is concave towards the axis major.

PROPOSITION IX.

The rectangle under the segments of the axis made by any ordinate is to the square of the ordinate in a constant ratio. (See Fig. 12.)

$$PM^2 : QM^2 = CB^2 : CA^2, \quad \text{(Prop. VIII.)}$$

but $\quad QM^2 = A'M \times MA; \quad$ (35 III. Euclid.)

$$\therefore PM^2 : A'M \times MA = CB^2 : CA^2.$$

Cor. 1.—The Latus rectum is a third proportional to the axis major and axis minor. (See Fig. 12.)

For
$$\left(\frac{L}{2}\right)^2 : AF \times FA' = CB^2 : CA^2$$

$$\left(\frac{L}{2}\right)^2 : CB^2 = CB^2 : CA ; \quad (Cor.\ 3,\ \text{Prop. 11.})$$

hence
$$CA : CB = CB : \frac{L}{2}.$$

Cor. 2.—If from the foot of any ordinate a line be drawn parallel to the line joining the extremities of the axes, the square of the ordinate will be equal to the rectangle under the segments of the axis minor.
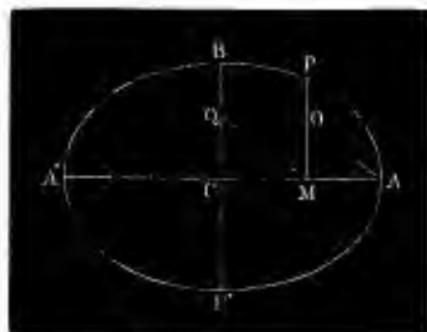
Fig. 14.

$$CA^2 : CM^2 = CB^2 : CQ^2 ; \quad \text{(Parallel lines.)}$$

$$\therefore CA^2 : CA^2 - CM^2 = CB^2 : CB^2 - CQ^2 ;$$

$$\therefore CA^2 : AM \times MA' = CB^2 : BQ \times QB' ;$$

$$\therefore CB^2 : PM^2 = CB^2 : BQ \times QB' ;$$

$$\therefore \qquad PM^2 = BQ \times QB'.$$

Cor. 3.—If *PN* be a normal at point *P*, meeting the axis major in *N*, and *NH* a perpendicular let fall on the focal radius vector, then *PH* is equal to half the Latus rectum.
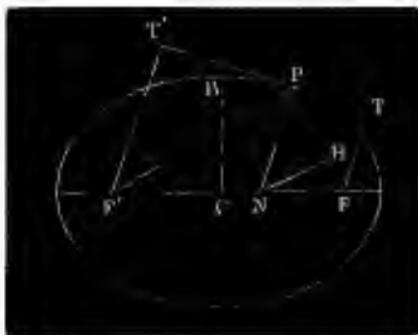
Fig. 15.

For $\qquad FT : FP = PH : PN.$ (Similar Δ^s.)

Also $\qquad F'T' : F'P = PH : PN;$ (Similar Δ^s.)

$$\therefore FT \times F'T' : FP \times F'P = PH^2 : PN^2$$

$$CB^2 : FP \times F'P = PH^2 : PN^2;$$ (Prop. v.)

but $\quad CA^2 : FP \times F'P = CB^2 : PN^2;$ (*Cor.* 2, Prop. vii.)

$$\therefore CA : CB = CB : PH;$$

$$\therefore PH = \tfrac{1}{2} \text{Latus rectum.} \qquad (Cor. 1, \text{Prop. ix.})$$

Cor. 4.—The normal and the focal perpendiculars on the tangent at any point are in Harmonic proportion.

Let the tangent *T'T*, when produced, meet the axis in *Q*.

Then $\qquad T'Q : QT = F'T' : FT.$ (Similar Δ^s.)

Also $\qquad F'T' : FT = T'P : PT;$ (Similar Δ^s.)

$$\therefore T'Q : QT = T'P : PT;$$

$\therefore$ *T'Q*, *PQ*, and *TQ*; and $\therefore$ *F'T'*, *NP*, and *FT*, are in Harmonic proportion.

PROPOSITION X.

The locus of the intersection of any tangent to an ellipse, with the line drawn through the focus perpendicular to the radius vector drawn from the focus to the point of contact of the tangent, is the directrix.
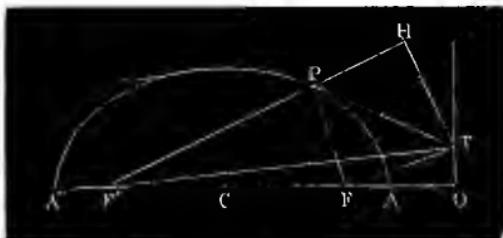
Fig. 16.

Let PT be any tangent, $FT \perp FP$.

Draw $TH \perp F'P$ produced, and $TO \perp F'F$ produced; join TF'.

The $\triangle^s$ TFP and THP are equal; (26 I. Euclid.)

$$\therefore FT = TH \text{ and } PF = PH;$$

$$\therefore F'H = F'P + PF = AA'.$$

Now $\quad F'H^2 = F'T^2 - TH^2 = F'T^2 - TF^2$

$$= F'O^2 - OF^2$$

$$= 2CO \times 2CF;$$

$$\therefore AA'^2 = 4CO \times CF, \text{ or } CA^2 = CO \times CF.$$

Hence OT is the directrix. (*Cor.* 4, Prop. I.)

Cor. 1.—Conversely, if from any point on the directrix a tangent be drawn to an ellipse, the line joining that point to the focus is perpendicular to the radius vector drawn from the focus to the point of contact of the tangent.

Cor. 2.—Tangents at the extremities of a focal chord intersect on the directrix.

PROPOSITION XI.

If any chord PP' of an ellipse cut the directrix in D, and if F be the focus corresponding to the directrix on which D is situated, then FD is the external bisector of the angle PFP'.

Fig. 17.

Produce $P'F$ to Q; draw Pp, $P'p'$ $\perp$ the directrix.

Then $\qquad P'F : PF = P'p' : Pp$

$$= P'D : PD; \quad \text{(Similar } \triangle^s.)$$

$\therefore$ FD bisects the $\angle PFG$. (Prop. B., VI. Euclid.)

Cor. 1.—Hence, being given one focus F, and three points P, P', P'', on an ellipse, we can find the directrix and the axes.

For, draw FD bisecting the $\angle$ between PF and $P'F$ produced : the point where this line meets $P'P$ produced will be one point on the directrix. Similarly, by bisecting the external $\angle$ between PF and $P''F$, another point on the directrix may be found, and hence the directrix itself.

Also, since the eccentricity is the ratio of PF to the perpendicular distance of P from the directrix, the axes may be found by Prop. I., and Cor.

F

If a tangent to a curve be defined as *The line joining two indefinitely near points on the curve,* it will follow immediately from this Prop. that *the right line drawn from the focus to the point of intersection of any tangent with the directrix is perpendicular to the radius vector drawn form the focus to the point of contact of the tangent;* also that *the tangent makes equal ∠ˢ with the radii vectores drawn from the foci to the point of contact.*

See Fig. 17.

For, when P is indefinitely near to P', PD becomes a tangent at the point P; and the $\angle PFP'$ is indefinitely small;

$$\therefore \; PFQ = 180°;$$

but $$\angle PFD = \tfrac{1}{2} \angle PFQ;$$

$$\therefore \; \angle PFD = 90°.$$
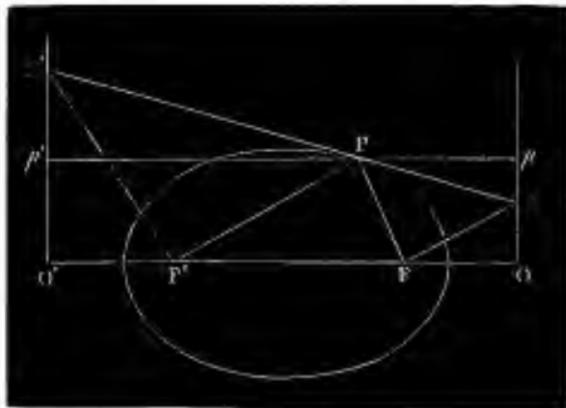
Fig. 18.

Again, $PF : PF' = Pp : Pp' = PD : PD'$,

(2, VI. Euclid.)

and $\angle PFD = \angle PF'D'$, each being 90°.

Hence $\triangle PFD$ is similar to $\triangle PF'D'$; (7, VI. Euclid.)

$$\therefore \; \angle FPD = \angle F'PD'.$$

PROPOSITION XII.

If two fixed points P, P', on an ellipse be joined with a third variable point O, the segment pp', intercepted on either directrix by the produced chords, subtends a constant angle at the focus corresponding to that directrix.

Fig. 19.

Since (by Prop. XI.) Fp is the bisector of the external $\angle$ between FO and FP;

$$\therefore \angle OFp + \tfrac{1}{2} \angle OFP = 90°.$$

Also Fp' is the bisector of the external $\angle$ between FO and FP';

$$\therefore \angle OFp' + \tfrac{1}{2} \angle OFP' = 90°;$$

$$\therefore \angle OFp + \tfrac{1}{2} \angle OFP = \angle OFp' + \tfrac{1}{2} \angle OFP';$$

$$\therefore \angle OFp - \angle OFp' = \tfrac{1}{2} (\angle OFP' - \angle OFP),$$

or $\qquad \angle pFp' = \tfrac{1}{2} \angle PFP'$, and $\therefore$ constant.

Cor. 1.—Hence the anharmonic ratio of the pencil formed by joining four fixed points on an ellipse to any fifth variable point is constant.

For $\qquad O \cdot PP'P''P''' = O \cdot pp'p''p''' = F \cdot pp'p''p'''.$

PROPOSITION XIII.

The line joining the focus to the intersection of two tangents to an ellipse bisect the angle which the points of contact subtend at the focus.
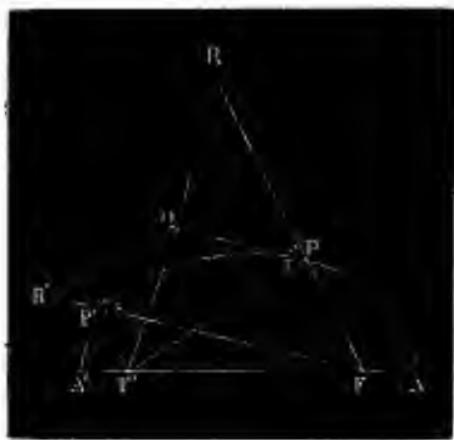
Fig. 20.

Let QP, QP', be tangents; join PF, PF', $P'F'$, $P'F$.

Produce FP till $PR = PF'$, and FP' till $P'R' = P'F'$; join QR, QR'.

The $\triangle^s$ $QF'P'$ and $QR'P'$ are equal; $\therefore QR' = QF'$.

(4, I. Euclid.)

The $\triangle^s$ QPF' and QPR are equal; $\therefore QR = QF'$;

(4, I. Euclid.)

hence $QR = QR'$.

But $FR' = FP' + P'F' = FP + PF' = FR$;

hence the $\triangle^s$ FQR and FQR' are equal; (8, I. Euclid.)

$$\therefore \angle QFR = \angle QFR'.$$

Cor. 1.—It is obvious from the above demonstration that

$$\angle QF'P' = \angle QR'P' = \angle QRP = \angle QF'P.$$

NOTE.—If the ordinate at any point P on an ellipse be produced to meet the Auxiliary circle in Q, then

$$\tan \tfrac{1}{2} PF'C = \sqrt{\frac{1-e}{1+e}} \cdot \tan \tfrac{1}{2} QCF; \text{ and } \tan \tfrac{1}{2} PFA = \sqrt{\frac{1+e}{1-e}} \tan \tfrac{1}{2} QCF.$$
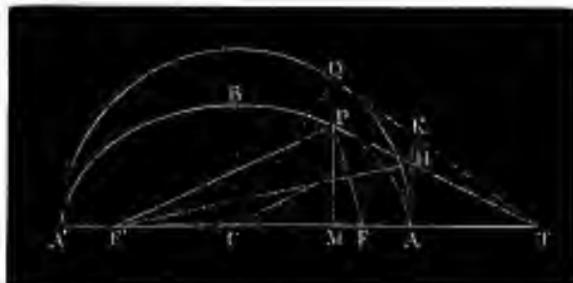
Fig. 21.

Draw the tangent at the vertex meeting the tangents at P and Q in H and K respectively; join KC, HF', HF. Let $CA = a$.

Then $\angle HF'C = \tfrac{1}{2} \angle PF'C$ and $\angle HFA = \tfrac{1}{2} \angle PFA$. (Prop. XIII.)

Also $\angle KCA = \tfrac{1}{2} \angle QCF$.

Now $\tan \tfrac{1}{2} PF'C = \tan HF'C = \dfrac{HA}{AF'},$

$\tan \tfrac{1}{2} PFA = \tan HFA = \dfrac{HA}{AF},$

$\tan \tfrac{1}{2} QCF = \tan KCA = \dfrac{KA}{AC};$

$\therefore \dfrac{\tan \tfrac{1}{2} PF'C}{\tan \tfrac{1}{2} QCF} = \dfrac{HA \times AC}{KA \times AF'} = \dfrac{PM}{QM} \cdot \dfrac{AC}{AF'} = \dfrac{CB}{CA} \cdot \dfrac{AC}{AF'}$ (Cor. 3, Prop. VIII.)

$$= \dfrac{CB}{AF'} = \dfrac{\sqrt{CA^2 - CF^2}}{CA + CF'} = \sqrt{\dfrac{CA - CF}{CA + CF}} = \sqrt{\dfrac{a - ae}{a + ae}};$$

$\therefore \tan \tfrac{1}{2} PF'C = \sqrt{\dfrac{1-e}{1+e}} \cdot \tan \tfrac{1}{2} QCF.$

Again, $\dfrac{\tan \tfrac{1}{2} PFA}{\tan \tfrac{1}{2} QCF} = \dfrac{HA \times AC}{KA \times AF} = \dfrac{PM}{QM} \cdot \dfrac{AC}{AF} = \dfrac{CB}{CA} \cdot \dfrac{AC}{AF}$ (Cor. 3, Prop. VIII.)

$$= \dfrac{CB}{AF} = \dfrac{\sqrt{a^2 - a^2 e^2}}{a - ae} = \sqrt{\dfrac{1+e}{1-e}};$$

$\therefore \tan \tfrac{1}{2} PFA = \sqrt{\dfrac{1+e}{1-e}} \cdot \tan \tfrac{1}{2} QCF.$

Cor. 1. $\dfrac{\tan \tfrac{1}{2} PF'C}{\tan \tfrac{1}{2} PFA} = \dfrac{1-e}{1+e}.$

Cor. 2.—If a quadrilateral be circumscribed to an ellipse, its opposite sides subtend supplementary angles at either focus.

Cor. 3.—By the aid of this Prop., also, we can draw a pair of tangents to an ellipse from any point without it.

With Q as centre and QF' as radius, describe a circle; with F as centre and a radius = the axis major, describe a circle; the lines joining the points of intersection of these circles with F will determine the points of contact of the tangents.

PROPOSITION XIV.

The angle at either focus, subtended by the segment intercepted on a variable tangent to an ellipse by two fixed tangents, is constant.
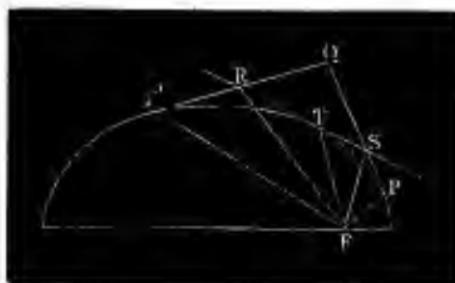
Fig. 22.

Let QP, QP', be the fixed tangents, RS the variable tangent; join FP', FR, FT, FS, FP.

Then $\angle TFS = \frac{1}{2} \angle TFP$, also $\angle TFR = \frac{1}{2} \angle TFP'$;

(Prop. xiii.)

$\therefore \angle RFS = \frac{1}{2} \angle P'FP$, and $\therefore$ constant.

Cor. 1.—The anharmonic ratio of the four points in which a variable tangent to an ellipse is cut by four fixed tangents is constant.

For the segments intercepted on the variable tangent subtend constant angles at the focus.

PROPOSITION XV.

If a tangent at the extremity of any diameter CP meet the axis produced in T, then the area of the $\triangle\ CPT$ thus formed will be equal to the area of the $\triangle\ CAH$, formed by producing the diameter CP to meet the tangent at the vertex A.
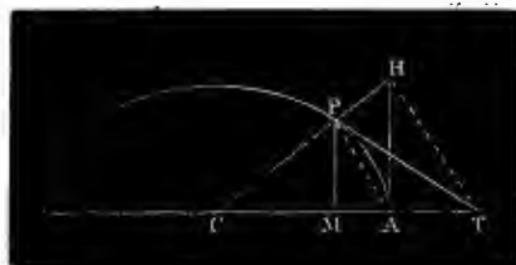
Fig. 23.

Draw the ordinate PM; join AP, HT.

Then $TC : CA = CA : CM$ (Prop. VIII.)

$= CH : CP$;   (Parallel lines.)

$\therefore\ PA$ is $\parallel HT$;   (2, VI. Euclid.)

$\therefore\ \triangle\ PHA = \triangle\ PTA$;   (37, I. Euclid.)

$\therefore\ \triangle\ CHA = \triangle\ CPT$.

Cor. 1.— $\triangle\ PMT$ = trap. $PMAH$.

Def.—The chords which join the extremities of any diameter to any point on the curve are called *supplemental chords.*

The axis major is sometimes called the *Transverse axis,* and the axis minor the *Conjugate axis.*

NOTE.—Throughout the following Props. CA denotes the semi-axis major, and CB the semi-axis minor.

Proposition XVI.

If an ordinate, drawn to the axis from any point P on an ellipse, be produced to meet the auxiliary circle in Q, and CQ' be drawn perpendicular to CQ, the perpendicular drawn from Q' to the axis will meet the ellipse in a point P', through which, if a diameter be drawn, it will be parallel to the tangent at P.

Fig. 24.

The tangent to the circle at Q will intersect the tangent to the ellipse at P on the axis.   (*Cor.* 2, Prop. VIII.)

Then, since QT and $Q'C$ are both $\perp CQ$; $\therefore$ they are parallel.

Hence $\triangle^s QMT$ and $Q'TM'$ are similar;

$\therefore MT : M'C = QM : Q'M'$ (Similar $\triangle^s$.)

$\qquad\qquad\qquad = PM : P'M'.$ (*Cor.* 3, Prop. VIII.)

Hence $\triangle^s PMT$ and $P'M'C$ are similar;

$\therefore P'C$ is $\parallel PT.$

Def.—The diameter which is parallel to the tangent at any point is said to be *conjugate* to the diameter which passes through that point.

Cor. 1.—If we draw tangents at P' and Q', it can be similarly shown that CP is parallel to the tangent at P': hence, if the diameter which passes through P' be conjugate to that through P; conversely, the diameter which passes through P will be conjugate to that through P'.

Cor. 2. $CM = Q'M'$, and $CM' = QM.$

Since $\angle M'Q'C + \angle M'CQ' = 90°,$

and also $\angle M'CQ' + \angle MCQ = 90° ;$

$$\therefore \angle M'Q'C = \angle MCQ.$$

Hence the $\triangle^s M'Q'C$ and MCQ are similar;

but $CQ' = CQ ; \therefore CM = QM'$, and $CM' = QM.$

Cor. 3. $CM^2 + CM'^2 = CA^2.$

For $CA^2 = CQ^2 = CM^2 + QM^2 = CM^2 + CM'^2.$ (*Cor.* 2.)

Cor. 4. $PM^2 + P'M'^2 = CB^2.$

Since $QM : Q'M' = PM : P'M' ;$ (*Cor.* 3, Prop. VIII.)

$$\therefore QM^2 + Q'M'^2 : QM^2 = PM^2 + P'M'^2 : PM^2.$$

Alt. $\therefore CA^2 : PM^2 + P'M'^2 = QM^2 : PM^2$ (*Cors.* 2 & 3.)

$$= CA^2 : CB^2 ;$$

(*Cor.* 3, Prop. VIII.)

$$\therefore PM^2 + P'M'^2 = CB^2.$$

Cor. 5.—The sum of the squares of any pair of semi-conjugate diameters is equal to the sum of the squares of the semi-axes.

For $CM^2 + CM'^2 = CA^2,$ (*Cor.* 2.)

and $PM^2 + P'M'^2 = CB^2 ;$ (*Cor.* 3.)

$$\therefore CP^2 + CP'^2 = CA^2 + CB^2.$$

Cor. 6. $PM : CM' = CB : CA : : P'M' : CM.$

For $PM : QM = CB : CA ;$ but $QM = CM' ;$

$$\therefore PM : CM' = CB : CA.$$

Again $P'M' : Q'M' = CB : CA ;$ but $Q'M' = CM ;$

$$\therefore P'M' : CM = CB : CA.$$

Cor. 7.—The $\triangle^s CPM$ and $CP'M'$ are equal in area.

For $PM : P'M = QM : Q'M' = CM' : CM ;$ (*Cor.* 2.)

$$\therefore PM \times CM = P'M' \times CM'.$$

Proposition XVII.

If QG be an ordinate drawn to any diameter Pp from any point Q on an ellipse, cutting the transverse axis in L, QS an ordinate drawn to the axis from the same point, and produced, if necessary, to meet the diameter Pp in K; AH a tangent at the vertex: then $\triangle QLS$ = trap. $KSAH$.
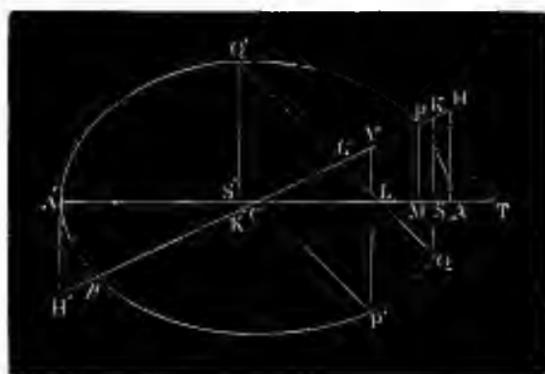
Fig. 25.

Draw the ordinate PM to the axis.

Then $\quad CA : CS : CM = AH : SK : MP$;

$$\text{(Similar } \triangle^s.)$$

$\therefore CA + CS : CA + CM = AH + SK : AH + MP,$

or $\qquad A'S : A'M = AH + SK : AH + MP$;

$\therefore A'S \times AS : A'M \times AM = (AH + SK) \, AS :$
$$(AH + MP) \, AM;$$

$\therefore SQ^2 : PM^2 =$ trap. $KSAH$: trap. $PMAH$;

$$\text{(Prop. ix.)}$$

$\therefore \triangle QSL : \triangle PMT =$ trap. $KSAH$: trap. $PMAH$;

$$\text{(19, VI. Euclid.)}$$

but $\qquad \triangle PMT =$ trap. $PMAH$; (*Cor.* 1, Prop. xv.)

$\therefore \triangle QSL =$ trap. $KSAH$.

Cor. 1.— $\triangle\ QGK$ = trap. $TPGL.$

For $\triangle\ QSL$ = trap. $KSAH$;

$\therefore$ $\triangle\ QGK$ = trap. $GLAH$ = trap. $TPGL.$

Cor. 2.—In like manner it can be shown that

$$\triangle\ GQ'K' = \text{trap. } TPGL \ ;$$

$$\therefore\ \ \triangle\ GQK = \triangle\ GQ'K' \ ;$$

but they are also similar ;

$$\therefore\ \ GQ = GQ'.$$

Hence any diameter bisects all chords parallel to the tangents at its extremities.

PROPOSITION XVIII.

If CP' be the semi-diameter conjugate to CP, QG any ordinate to CP ; then $QG^2 : PG \times Gp = CP'^2 : CP^2$.

For $\triangle\ CPT : \triangle\ CGL = CP^2 : CG^2$.

Division. $\triangle\ CPT :$ trap. $TPGL = CP^2 : CP^2 - CG^2$,

or $\triangle\ CP'V : \triangle\ QGK = CP^2 : PG \times Gp$;

                              (*Cor.* 1, Prop. xviii.)

$$\therefore\ \ CP'^2 : GQ^2 = CP^2 : PG \times Gp,$$

                              (19, VI. Euclid.)

or $QG^2 : PG \times Gp = CP'^2 : CP^2$.

Cor. 1.—Ordinates to any diameter at equal distances from the centre are equal.

For the rectangles are the same for equal distances from the centre.

Cor. 2.—The squares of the ordinates to any diameter are proportional to the rectangles under the segments of the diameter.

Proposition XIX.

If a normal at any point P of an ellipse meet the axis major in N, and the axis minor in N'; then

$$PN : CP' :: CB : CA :: CP' : PN',$$

CP' being the semi-diameter conjugate to CP.
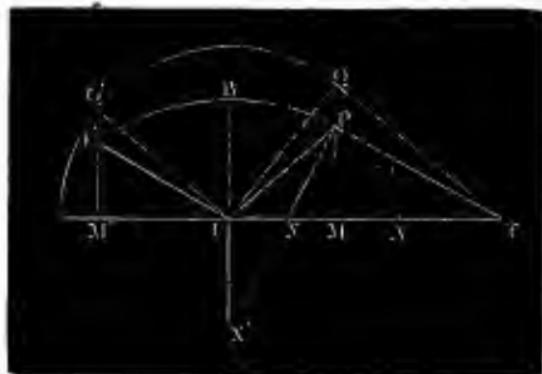
Fig. 26.

Since CP' is $\parallel TP$, the $\triangle P'CM'$ is similar to the $\triangle PMT$, and $\therefore$ similar to $\triangle NPM$. (8, VI. Euclid.)

$$\therefore P'C : NP = CM' : PM$$

$$= QM : PM \ (Cor. 2, Prop. \text{xvi.})$$

$$= CA : CB. \ (Cor. 2, Prop. \text{viii.})$$

Also $\qquad PN' : PN = CM : MN.$ $\qquad$ (Similar $\triangle^s$.)

$$= CA^2 : CB^2; \qquad (Prop. \text{vii.})$$

$$\therefore PN' \times PN : PN^2 = CA^2 : CB^2$$

$$= P'C^2 : PN^2;$$

$$\therefore PN' \times PN = P'C^2;$$

$$\therefore PN' : P'C = P'C : PN$$

$$= CA : CB.$$

Cor. $\qquad PN \times PN' = P'C^2.$

PROPOSITION XX.

The rectangle under the distances of the foci, from any point P on an ellipse, is equal to the square of the semi-diameter CP', conjugate to that passing through the point.
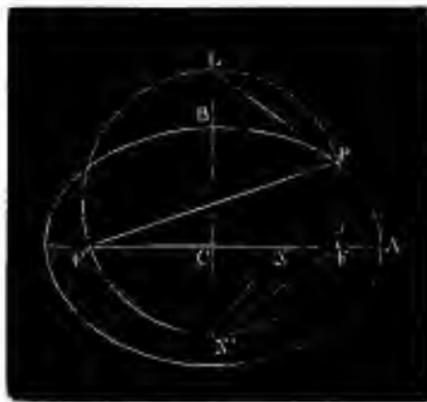
Fig. 27.

Circumscribe the $\triangle F'PF$ by a circle cutting the axis minor in N' and L; join PN' and $N'F$.

Since CL bisects FF', and is also $\perp$ it; $\therefore$ LN' is the diameter of the circle, and arc $F'N' = $ arc FN';

$$\therefore \angle F'PN' = \angle FPN'. \quad (26, \text{III. Euclid.})$$

Also $\quad \angle PF'N = \angle PN'F;$

hence the $\triangle^s F'PN$ and $N'PF$ are similar;

$$\therefore F'P : PN = N'P : PF;$$

$$\therefore F'P \times PF = PN \times N'P$$

$$= CP'^2. \quad (Cor. 1, \text{Prop. xix.})$$

Cor. 1.—$PN^2 = PN' \times PN - N'N \times NP$

$$(3, \text{II. Euclid.})$$

$$= CP'^2 - F'N \times NF.$$

Otherwise thus :—

$$2CA = PF + PF' ;$$
$$\therefore 4CA^2 = PF^2 + PF'^2 + 2PF \times PF' \quad (4, \text{ II. Euclid.})$$
$$= 2CP^2 + 2CF^2 + 2PF \times PF' ;$$
$$\therefore PF \times PF' = 2CA^2 - CF^2 - CP^2$$
$$= CA^2 + CB^2 - CP^2 \qquad (Cor. \; 3, \text{ Prop. ii.})$$
$$= CP^2 + CP'^2 - CP^2 \qquad (Cor. \; 3, \text{ Prop. xvi.})$$
$$= CP'^2.$$

Cor. 2.—Since PN' bisects the $\angle FPF'$ it is a normal, and $\therefore PL$, which is $\perp PN'$, is a tangent. Hence the circle which passes through the foci, and any point P on the ellipse, passes also through the points in which the tangent and normal at P intersect the axis minor.

Cor. 3.—If K be the foot of the perpendicular let fall from N' on either PF' or PF produced,

then $\qquad PK = \frac{1}{2}(PF + PF') = $ semi-axis major.

This follows immediately from the well-known properties of the triangle FPF', and its circumscribing circle. (See Geometrical Gymnasium, Exercise 179, Galbraith and Haughton's Manual of Euclid, books i., ii., iii.)

Cor. 4.—If K' be the foot of the perpendicular let fall from L on PF',

then $\qquad F'K' = \frac{1}{2}(PF + PF') = $ semi-axis major.

Also if K'' be the foot of the perpendicular let fall from L on FP produced,

then $\qquad FK'' = \frac{1}{2}(PF + PF') = $ semi-axis major.

(See as in *Cor.* 3.)

Cor. 5. $\qquad CA^2 : CB^2 = PF' \times PF : PN^2$
$$(Cor. \; 2, \text{ Prop. vii.})$$
$$= CP'^2 : PN^2 ;$$
$$\therefore CA : CB = CP' : PN.$$

PROPOSITION XXI.

If CK be a perpendicular let fall from the centre on a tangent to an ellipse at any point P, and CP' the semi-diameter conjugate to CP, then $CK : CA :: CB : CP'$.
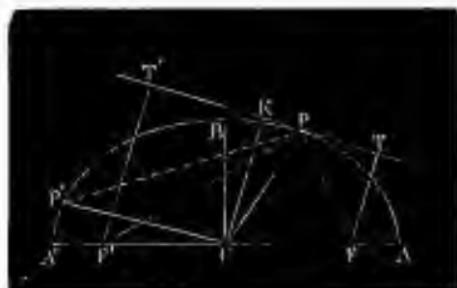
Fig. 28.

Draw $\quad FT, F'T' \perp$ the tangent at P.

Since the $\quad \triangle^s FPT$ and $F'PT'$ are similar;

$$\therefore FP : F'P = FT : F'T';$$
$$= FP + F'P : FT + F'T'$$
$$= CA : CK;$$
$$\therefore FP \times F'P : FT \times F'T' = CA^2 : CK^2;$$

but $\qquad\qquad FP \times F'P = CP'^2,$ (Prop. xx.)

and $\qquad\qquad FT \times F'T' = BC^2;$ (Prop. v.)

$$\therefore CP'^2 : BC^2 = CA^2 : CK^2;$$
$$\therefore CP' : BC = CA : CK.$$

Cor.—The area of the triangle formed by joining the extremities of any pair of conjugate semi-diameters is constant.

For $\qquad CP' \times CK = CB \times CA;$

$\therefore$ area of $\triangle P'CP =$ area of $\triangle BCA$, and $\therefore$ constant.

PROPOSITION XXII.

If two chords of an ellipse intersect one another, the rectangles under the segments are proportional to the squares of the parallel semi-diameters.

Fig. 29.

Let PP' be any chord drawn through O, and CR the parallel semi-diameter. Draw the ordinates PM, $P'M'$, RN to the axis, and produce them to meet the auxiliary circle in Q, Q', S.

Then $PM : P'M' = QM : Q'M'$; (*Cor.* 2, Prop. VIII.)

∴ PP' and QQ', if produced, will meet the axis produced in the same point T.

Through O draw $O'OL \perp$ the axis.

Again $NC : MT = RN : P'M'$ (Similar △ˢ.)
$$= SN : Q'M';$$
(*Cor.* 3, Prop. VIII.)

∴ SC is $\parallel Q'T$; hence △ SCR is similar to △ $Q'TP'$.

Now $OP' : O'Q' = P'T : Q'T$, (Parallel lines.)
and $OP : O'Q = P'T : Q'T$;

∴ $OP \times OP' : O'Q \times O'Q' = P'T^2 : Q'T^2$
$$= RC^2 : CS^2.$$

Alt. $OP \times OP' : CR^2 = O'Q \times O'Q' : CS^2.$

Similarly, if any other chord pp' be drawn through O, and the ordinates pm, $p'm'$ produced to meet the auxiliary circle in q, q' : then, since

$$OL : O'L = CB : CA$$
$$= pm : qm = p'm' : q'm';$$
$$(Cor. \text{ 3, Prop. VIII.})$$

$\therefore$ the chord qq' will pass through O';
and if Cr be the semi-diameter parallel to pp',

then $\qquad Op \times Op' : Cr^2 = O'q \times O'q' : CA^2$.

But $\qquad O'Q \times O'Q' = O'q \times O'q'$; $\qquad$ (35, III. Euclid.)

hence $\qquad OP \times OP' : Op \times Op' = CR^2 : Cr^2$.

$$(See \text{ Note, page } 83.)$$

Otherwise thus :—
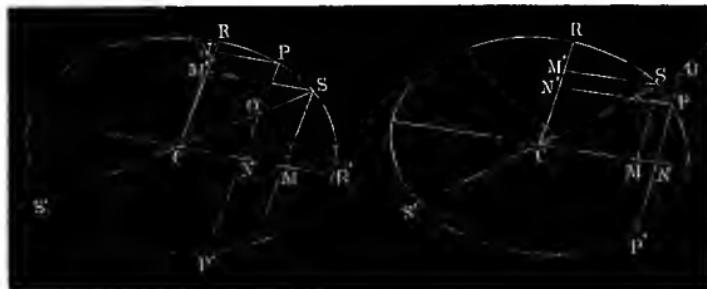
Fig. 30. $\qquad\qquad$ Fig. 31.

Let PP' be any chord drawn through the point O, and CR the parallel semi-diameter, CR' the semi-diameter conjugate to CR; draw the diameter through O, meeting the curve in S; draw SM an ordinate to CR', and SM', PN' ordinates to CR.

Then $\quad RC^2 - CN'^2 : RC^2 - CM'^2 = PN'^2 : SM'^2$;
$$(Cor. \text{ 2, Prop. XVIII.})$$
$$\therefore RC^2 - PN^2 : RC^2 - SM^2 = CN^2 : CM^2$$
$$= ON^2 : SM^2.$$
Alt. $\quad RC^2 - PN^2 : ON^2 = RC^2 - SM^2 : SM^2$.
Comp. $RC^2 - PN^2 + ON^2 : ON^2 = RC^2 : SM^2$.

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Alt. $RC^2 - PO \times OP' : RC^2 = ON^2 : SM^2$

$= OC^2 : CS^2.$

Div. $PO \times OP' : RC^2 = CS^2 - OC^2 : CS^2$

$= SO \times OS' : CS^2.$

If pp' be any other chord drawn through O, and Cr the parallel semi-diameter; then

$$pO \times Op' : Cr^2 = SO \times OS' : CS^2.$$

Hence $PO \times OP' : pO \times Op' = CR^2 : Cr^2.$

Cor. 1.—The rectangle under the segments of any diameter made by an ordinate is to the square of the ordinate as the square of the semi-diameter is to the square of its semi-conjugate.

This follows by supposing one of the chords to pass through the centre, and the other parallel to the tangent at its extremity.

Cor. 2.—If O be without the ellipse, and P, P' coincide, also p, p'; then the tangents to an ellipse from any point without it are proportional to the parallel semi-diameters.

Cor. 3.—If two parallel tangents OP, $O'P'$, be met by any third tangent, OO'; then $OP : O'P' = OQ : O'Q$.
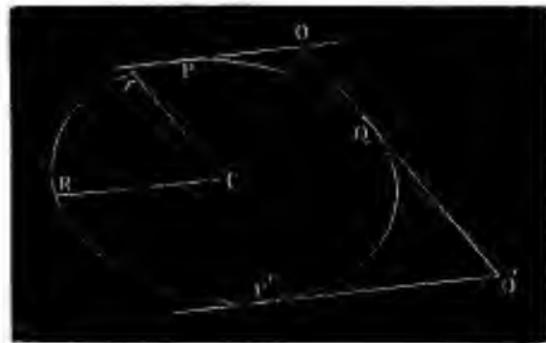
Fig. 32.

For $OP : OQ = CR : Cr = O'P' : O'Q.$ (*Cor.* 2.)

Cor. 4.—If from a point without an ellipse a secant and also a tangent be drawn, the rectangle under the whole secant and its external segment is to the square of the tangent as the squares of the parallel semi-diameters.

NOTE.—If for O, O', L, we read o, o', l, the demonstration, page 80, will apply to the case when the point in which the chords intersect lies without the ellipse.

PROPOSITION XXIII.

If from a point without an ellipse two tangents OP, OR be drawn, any line AQ' drawn parallel to either will be cut by the curve and chord of contact of the tangents in Geometric proportion.
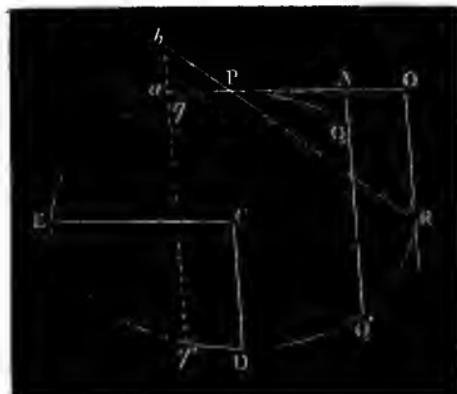
Fig. 33.

Draw the semi-diameters CE and $CD \parallel OP$ and OR respectively.

Then $\quad AQ \times AQ' : AP^2 = CD^2 : CE^2$
$$(Cor.\ 4,\ \text{Prop. XXII.})$$
$$= OR^2 : OP^2$$
$$(Cor.\ 2,\ \text{Prop. XXII.})$$
$$= AB^2 : AP^2 ;$$
$$(\text{Parallel lines.})$$
$$\therefore\ AQ \times AQ' = AB^2.$$

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Cor.—If AQ' be a tangent : then $AQ = AB$.

That is, if two parallel tangents to an ellipse be cut by any third tangent, the segment intercepted on either of the parallel tangents, between the chord of contact of the other two and the curve, is bisected by the non-parallel tangent.

PROPOSITION XXIV.

If a circle intersect an ellipse in four points, the common chords will be equally inclined to the axis.
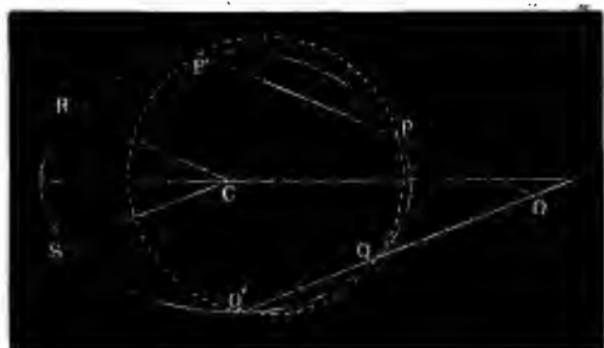
Fig. 34.

Let PP', QQ' be the common chords intersecting in O.
Draw the semi-diameters CR, $CS \parallel PP'$, QQ' respectively.

Then $OP \times OP' : OQ \times OQ' = CR^2 : CS^2$;
                                        (Prop. XXII.)

but $OP \times OP' = OQ \times OQ'$;   (35, III Euclid.)

$\therefore CR = CS$.

Hence CR and CS, and $\therefore OP'$ and OQ', are equally inclined to the axis.   (*Cor.* 1, Prop. I.)

Cor. 1.—In like manner it can be shown that PQ and $P'Q'$, also PQ' and $P'Q$ are equally inclined to the axis.

PROPOSITION XXV.

If a tangent to an ellipse at any point P meet any diameter AA' produced in T, and the ordinate PM be drawn; then CA is a mean proportional between CM and CT.
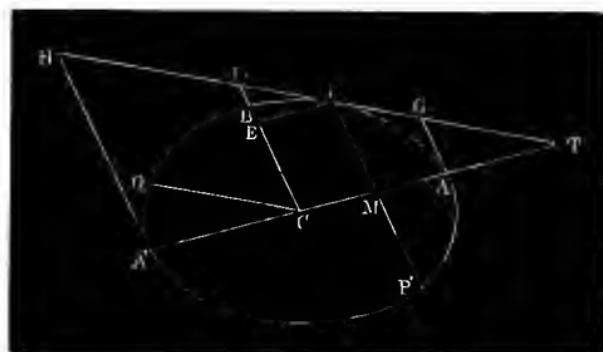
Fig. 35.

Draw tangents at A and A', meeting the tangent at P in G and H.

Then $A'T : TA = A'H : AG$;   (Parallel lines.)

and $A'M : MA = HP : PG$;

but $A'H : AG = HP : PG$;
                                        (*Cor.* 3, Prop. XXII.)

          $\therefore\ A'T : TA = A'M : MA$;

hence $CM \times CT = CA^2$.

Cor. 1.—Conversely, if PM be an ordinate to any diameter AA', and CT be taken a third proportional to CM and CA; then PT will be a tangent to the ellipse.

Cor. 2.—The tangents at the extremities of any double ordinate intersect on the diameter corresponding to that ordinate.

Cor. 3.—If a diameter be drawn through the intersection of two tangents to an ellipse, it will bisect the chord of contact.

Proposition XXVI.

If a variable tangent to an ellipse meet two fixed parallel tangents, it will intercept segments on them whose rectangle is constant, and equal to the square of the parallel semi-diameter.

(See Fig. 35.)

Let GPH be the variable tangent meeting the fixed tangents AG, $A'H$ in the points G, H.

Draw PE an ordinate to BC.

Then $\qquad CT : CA = CA : CM.$ $\qquad$ (Prop. xxv.)

Division $\qquad CT : TA = CA : AM.$

Alt. $\qquad CT : CA = TA : AM.$

Comp. $\qquad CT : TA' = TA : TM;$

$\qquad \therefore CD : A'H = AG : PM;$ $\qquad$ (Similar $\triangle^s$.)

$\qquad \therefore A'H \times AG = CD \times PM$

$\qquad\qquad\qquad = CD \times CE = CB^2.$ (Prop. xxv.)

Cor. 1.—The rectangle under the segments of the variable tangent is equal to the square of the semi-diameter CQ drawn parallel to it.

For $\qquad A'H : HP = CB : CQ = AG : PG ;$
$\qquad\qquad\qquad\qquad\qquad$ (*Cor.* 3, Prop. xxii.)

$\qquad \therefore A'H \times AG : HP \times PG = CB^2 : CQ^2.$

But $\qquad\qquad\qquad A'H \times AG = CB^2;$

$\qquad \therefore HP \times PG = CQ^2.$

Cor. 2.—Any tangent to an ellipse will be cut harmonically by two parallel tangents and the diameter passing through their points of contact.

For $\qquad\qquad HT : TG = A'H : AG \qquad$ (Similar $\triangle^s$.)

$$= HP : PG.$$
$$(Cor. 3, \text{Prop. xxii.})$$

Cor. 3.—If two parallel tangents AC, BF, be cut by any two other tangents EF, CD; then

$$AE : BD = EO : OF = CO : OD.$$
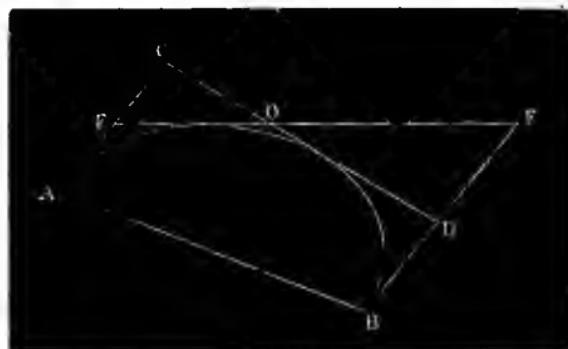

Fig. 36.

For $\qquad CA \times BD = AE \times BF; \qquad$ (Prop. xxvi.)

$$\therefore AE : AC = BD : BF;$$

$$\therefore AE : AC - AE = BD : BF - BD.$$

Alt. $\qquad AE : BD = EC : DF$

$$= EO : OF, \text{ or } = CO : OD.$$

Cor. 4.—The lines joining F with C, and D with E, would, if produced, intersect on the diameter BA produced.

PROPOSITION XXVII.

The triangles CPT and CAK, formed by drawing tangents at the extremities of any two semi-diameters of an ellipse, are equal in area.

Fig. 37.

Draw the ordinate PM.

Then

$$TC : CA = CA : CM. \qquad \text{(Prop. xxv.)}$$
$$= CK : CP; \qquad \text{(Parallel lines.)}$$
$$\therefore KT \text{ is } \| PA ;$$
$$\therefore \triangle PTA = \triangle PKA ; \qquad \text{(37, I. Euclid.)}$$
$$\therefore \triangle CPT = \triangle CAK.$$

Cor. 1.—If the ordinate AN be drawn from A to the semi-diameter CP, then the area of the $\triangle CMP$ = area of $\triangle CAN$.

For

$$AN \text{ is } \| PT ;$$
$$\therefore TC : CA = PC : CN ;$$

hence

$$CA : CM = PC : CN ;$$
$$\therefore MN \text{ is } \| PA ;$$
$$\therefore \triangle NPM = \triangle NAM ; \qquad \text{(37, I. Euclid.)}$$
$$\therefore \triangle CMP = \triangle CAN.$$

Proposition XXVIII.

If from the extremities of any two conjugate diameters CP, CQ, the ordinates PM, QN be drawn to any other diameter CA ; then

$$CN^2 = AM \times MA', \text{ and } CM^2 = AN \times NA'.$$
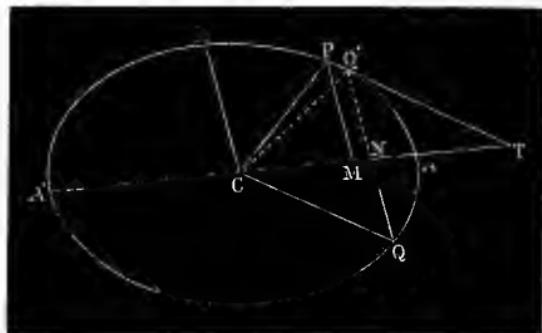
Fig. 38.

$$MT^2 : CN^2 = MP^2 : QN^2 \qquad \text{(Similar } \triangle^s.)$$
$$= MA \times MA' : NA \times NA'.$$
$$\text{(\textit{Cor.} 2, Prop. xviii.)}$$

But $\quad MA \times MA' = AC^2 - CM^2 \qquad$ (5, II. Euclid.)

$$= TC \times CM - CM^2 \qquad \text{(Prop. xxv.)}$$
$$= CM \times MT ;$$

$$\therefore \ MT^2 : CN^2 = CM \times MT : AN \times NA'.$$

Alt. $\quad MT^2 : CM \times MT = CN^2 : CA^2 - CN^2 ;$
$$\text{(5, II. Euclid.)}$$

$$\therefore \ MT : CM = CN^2 : CA^2 - CN^2.$$

Comp. $\quad MT : CT = CN^2 : CA^2 ;$

$$\therefore \ CM \times MT : CM \times CT = CN^2 : CA^2.$$

But $CM \times CT = CA^2;$ (Prop. xxv.)

$\therefore CN^2 = CM \times MT = AM \times MA'.$

In like manner it can be proved that $CM^2 = AN \times NA'.$

Cor. 1.— $CM^2 + CN^2 = CA^2.$

For $CM^2 + CN^2 = AN \times NA' + AM \times MA'$

$$= CA^2 - CN^2 + CA^2 - CM^2;$$
(5, II. Euclid.)

$\therefore 2CM^2 + 2CN^2 = 2CA^2;$

or, $CM^2 + CN^2 = CA^2.$

Cor. 2.—$PM^2 + QN^2 = BC^2,$ BC being the semi-diameter conjugate to $CA.$

For $CB^2 : CA^2 = PM^2 : AM \times MA'$ or $CN^2;$
(Prop. xviii.)

also, $CB^2 : CA^2 = QN^2 : AN \times NA'$ or $CM^2;$

$\therefore CB^2 : CA^2 = PM^2 + QN^2 : CN^2 + CM^2.$

But $CA^2 = CN^2 + CM^2, \therefore CB^2 = PM^2 + QN^2.$

Cor. 3.—$CA : CB = CM : QN = CN : PM.$

For $CA^2 : CB^2 = AM \times MA',$ or $CN^2 : PM^2$

$$= AN \times NA',$ or $CM^2 : QN^2;$$

$\therefore CA : CB = CN : PM = CM : QN.$

Cor. 4.—The $\triangle CMP$ is $= \triangle CNQ$ in area.

Produce QN to $Q'.$ Then $QN = Q'N;$

$\therefore CN : CM = PM : Q'N;$ (*Cor.* 3.)

$\therefore \triangle CMP = \triangle CNQ' = \triangle CNQ.$
(15, VI. Euclid.)

PROPOSITION XXIX.

If any tangent to an ellipse meet any two conjugate diameters CP, CQ, the rectangle under its segments is equal to the square of the parallel semi-diameter.
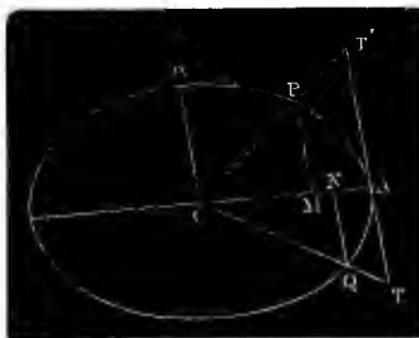
Fig. 39.

Draw the ordinates, PM, QN, to the diameter passing through the point of contact of the tangent.

Then $\qquad CM : PM = CA : AT'$; $\qquad$ (Similar Δ^s.)

$\qquad \therefore CM \times AT' = PM \times CA$

$\qquad \qquad \qquad = CB \times CN$;
$\qquad \qquad \qquad \qquad$ (*Cor.* 3, Prop. xxviii.)

$\qquad \therefore CM : CN = CB : AT'$.

Again, $\qquad CN : NQ = CA : AT$; $\qquad$ (Similar Δ^s.)

$\qquad \therefore CN \times AT = CA \times NQ$

$\qquad \qquad \qquad = CB \times CM$;
$\qquad \qquad \qquad \qquad$ (*Cor.* 3, Prop. xxviii.)

$\qquad \therefore CM : CN = AT : CB$;

hence $CB : AT' = AT : CB$;

$$\therefore AT \times AT' = CB^2.$$

Otherwise thus :—
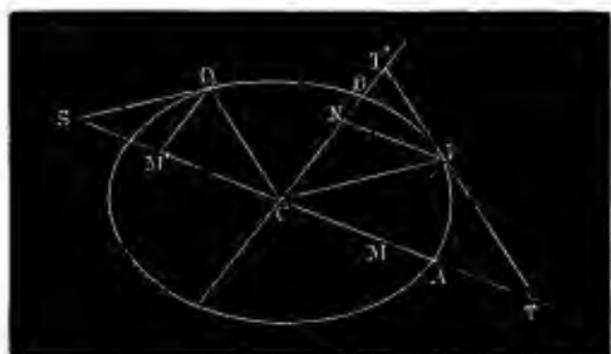
Fig. 40.

Draw PM, QM' ordinates to CA, and PN an ordinate to CB.

Then $CS \times CM' = CA^2 = CM \times CT$;   (Prop. xxv.)

$$\therefore CS : CT = CM : CM'.$$

But $CS : CT = CQ : PT$;   (Similar Δ^s.)

$$\therefore CQ : PT = CM : CM'$$

$$= PN : CM'$$

$$= PT' : CQ;   \text{(Similar } \Delta^s\text{.)}$$

$$\therefore PT \times PT' = CQ^2.$$

Cor.—Conversely, if on any tangent segments AT, AT' be measured from the point of contact whose rectangle is equal to the square of the parallel semi-diameter, then the diameters drawn through T and T' will be conjugate.

PROPOSITION XXX.

Given in magnitude and position any two conjugate semi-diameters CA', CB', of an ellipse, to find the axes.
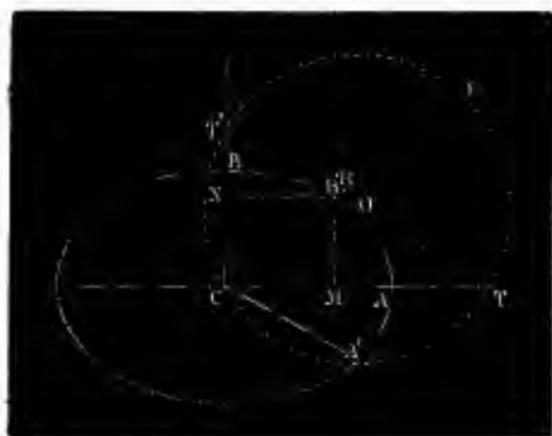
Fig. 41.

On CB' produced take BD a third proportional to CB' and CA'; bisect CD in H; draw $HO \perp CD$ to meet $B'O$ drawn $\parallel CA'$; with the centre O and radius OC describe a circle cutting $B'O$ produced in T and T'; join CT, CT'. Draw $B'M$ and $B'N \parallel CT'$ and CT respectively. Take CA a mean proportional between CT and CM; also CB a mean proportional between CT' and CN. Then CA and CB are the semi-axes.

For $\qquad CA'^2 = CB' \times B'D \qquad$ (Const.)

$\qquad\qquad\quad = TB' \times B'T'; \qquad$ (35, III. Euclid.)

$\qquad \therefore\ CA$ and CB are conjugate diameters;

$\qquad\qquad\qquad\qquad\qquad\qquad$ (*Cor.*, Prop. XXIX.)

but $\angle\ TCT' = 90°$; hence CA and CB are the semi-axes.

PROPOSITION XXXI.

If a straight line cut two tangents TP, TQ to an ellipse in the points A, A', the curve in B, B', and the chord of contact in O; then

$$AO^2 : A'O^2 = AB \times AB' : A'B \times A'B'.$$

Fig. 42.　　　　　　　　　Fig. 43.

Draw $AM \parallel TQ$; also the semi-diameters CH, CK, and $CL \parallel TP$, TQ and AA', respectively.

Then $AO^2 : A'O^2 = AM^2 : A'Q^2$ (Parallel lines.)

$$= \begin{Bmatrix} AM^2 : AP^2 \\ AP^2 : A'Q^2 \end{Bmatrix}$$

$$= \begin{Bmatrix} TQ^2 : TP^2 \\ AP^2 : A'Q^2 \end{Bmatrix} \quad \text{(Similar } \Delta^s.)$$

$$= \begin{Bmatrix} CK^2 : CH^2 \\ AP^2 : A'Q^2 \end{Bmatrix}$$
$$\text{(Cor. 2, Prop. xxii.)}$$

$$= \begin{bmatrix} CK^2 : CL^2 \\ CL^2 : CH^2 \\ AP^2 : A'Q^2 \end{bmatrix}$$

$$= \left\{ \begin{array}{l} A'Q^2 : A'B \times A'B' \\ AB \times AB' : AP^2 \\ AP^2 : A'Q^2; \end{array} \right\}$$

(*Cor.* 4, Prop. XXII.)

$$\therefore AO^2 : A'O^2 = AB \times AB' : A'B \times A'B'.$$

PROPOSITION XXXII.

If any line TS be drawn parallel to the chord of contact of the tangents to an ellipse, the segments AT, BS intercepted between the curve and the tangents will be equal.
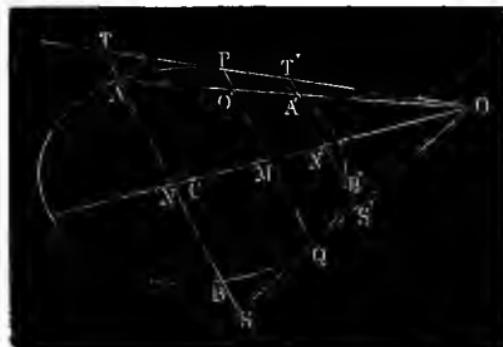
Fig. 44.

Draw the diameter bisecting the parallel chords PQ, AB. This diameter will pass through O.

(*Cor.* 2, Prop. XXV.)

Then, since $\qquad PM = MQ$;

$$\therefore TN = NS;$$

but $\qquad AN = NB$;

$$\therefore AT = BS.$$

PROPOSITION XXXIII.

Any line *OB*, drawn through the intersection of two tangents to an ellipse, is cut harmonically by the curve and the chord of contact of the tangents.
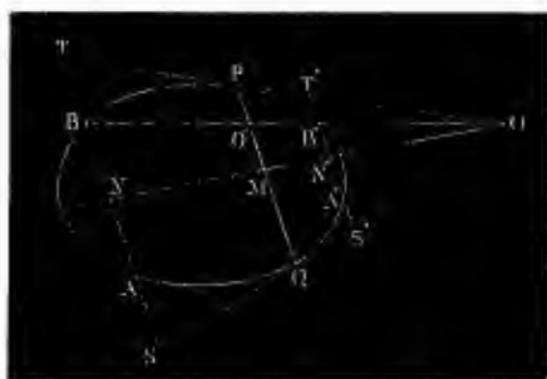
Fig. 45.

Through *B* and *B'* draw *TS* and *T'S'* ∥ *PQ*, the chord of contact of the tangents.

Then $\quad TP^2 : T'P^2 = TB \times TA : T'B' \times T'A'$;

$$\qquad\qquad (Cor.\ 4,\ \text{Prop. XXII.})$$

but $\qquad TB = AS$, and $T'B' = A'S'$; (Prop. XXXII.)

$\therefore TP^2 : T'P^2 = TB \times BS : T'B' \times B'S'$

$$\qquad\qquad = OB^2 : OB'^2 ; \qquad (\text{Similar } \triangle^s.)$$

but $\qquad TP : T'P = O'B : O'B'$; (Parallel lines.)

hence $\qquad OB : OB' = O'B : O'B'$.

Cor.—If through any point *O* without an ellipse a line be drawn cutting the curve in *B*, *B'*, and *OO'* be taken equal to the harmonic mean between *OB* and *OB'*, the locus of *O'* is a straight line; namely, the chord of contact of the tangents drawn from *O*.

PROPOSITION XXXIV.

Any line drawn through the middle point of the chord of contact of two tangents to an ellipse, will be cut harmonically by the curve, and the line drawn through the intersection of the tangents parallel to their chord of contact.
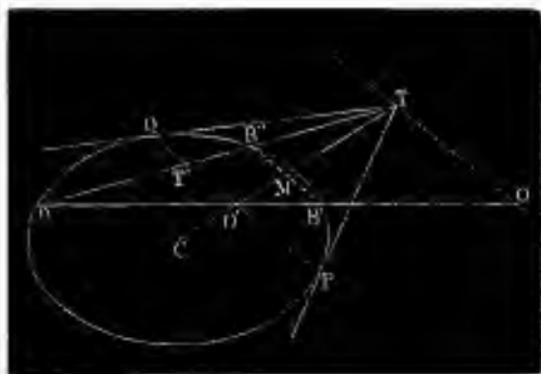
Fig. 46.

Let TP, TQ be tangents, BB' any chord drawn through O', the middle point of PQ, and meeting the line drawn through $T \parallel PQ$ in O. Through B'' draw $B'B'' \parallel PQ$; join TC.

Then TC will pass through O'.   (*Cor.* 3, Prop. xxv.)

$$B''M : B''B' = \begin{Bmatrix} B''M : O'T' \\ O'T' : B'B' \end{Bmatrix}$$

$$= \begin{Bmatrix} B''T : TT' \\ BT' : BB'' \end{Bmatrix} \quad \text{(Similar } \triangle\text{'s.)}$$

$$= B''T \times BT' : TT' \times BB''.$$

But $TT' \times BB'' = 2B''T \times BT'$;

$\therefore B'B'' = 2B''M$; $\therefore B'M$ is an ordinate, and B' a point on the curve.

Now $BT : TB'' = BT' : T'B''$;   (Prop. xxxiii.)

$\therefore BO : OB' = BO' : O'B'$.   (Parallel lines.)

H

Otherwise thus :—

Let TP, TQ be tangents, BB' any chord drawn through O', the middle point of PQ, and produced to meet the line drawn through $T \parallel PQ$ in O; join OC, TC; then (*Cor.* 3, Prop. xxv.) TC will pass through O'. Draw RN a tangent at R, $RM \parallel QP$, and $\therefore$ an ordinate to CT'; through O' draw $DE \parallel RN$.
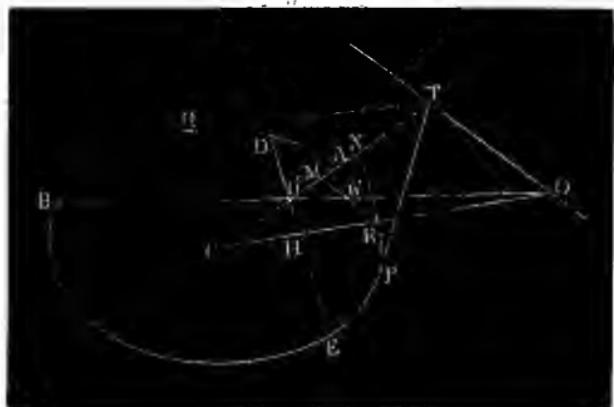
Fig. 47.

Then, $CT \times CO' = CA^2 = CN \times CM$; (Prop. xxv.)

$\therefore CT : CM = CN : CO'$;

$\therefore CO : CR = CR : CH$; (Parallel lines.)

$\therefore$ the tangents at D and E will intersect at O ;

(Prop. xxv.)

and hence $BO : OB' = BO' : O'B'$.

(Prop. xxxiii.)

Cor. 1.—If through any point O' within an ellipse any line be drawn cutting the curve in B, B', and O be taken the harmonic conjugate to O' with regard to B and B', the locus of O is a straight line.

Cor. 2.—If from O tangents be drawn to the ellipse, the chord of contact will pass through O'.

For (Prop. xxxiii.) the chord of contact must divide the line OB harmonically.

PROPOSITION XXXV.

If through any point P on an ellipse lines POO', $Q'PQ$ be drawn parallel to any two adjacent sides DA, DC of an inscribed quadrilateral, meeting the opposite sides in O, O', and Q, Q'; then $PO \times PO' : PQ \times PQ'$ in a constant ratio.
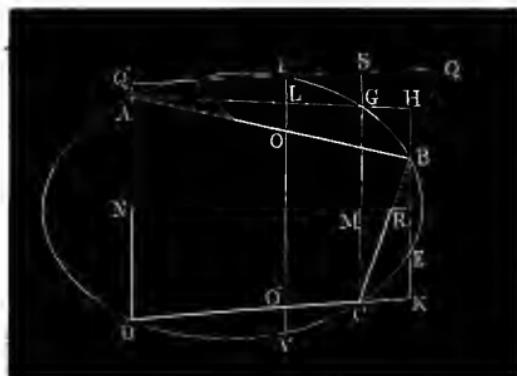
Fig. 48.

Through B and C draw BE and $CG \parallel AD$; join AG, and produce it to meet EB produced in H; draw the diameter NMR bisecting the parallel chords AD, GC, BE.

Then $\qquad HR = RK;$ (Parallel lines.)

hence $\qquad BH = EK$

Now $\qquad OL : HB = LA : AH$

$\qquad\qquad\qquad = O'D : DK.$

Alt. $\qquad OL : O'D = HB : DK.$

Also $P'O$ or $SC : QS = BK : KC.$ (Similar $\triangle^s$.)

Compounding,

$$OL \times PO' : O'D \times QS = HB \times BK : DK \times KC;$$

H 2

$$\therefore OL \times PO' : PQ' \times QS = EK \times BK : DK \times KC$$
$$= O'V \times O'P : O'C \times O'D$$
$$= PL \times O'P : PS \times PQ';$$
$$\therefore OL \times PO' + PL \times PO' : PQ' \times QS + PS \times PQ'$$
$$= EK \times BK : DK \times KC,$$

or $\quad PO \times PO' : PQ \times PQ' = EK \times BK : DK \times KC.$

Now it is evident that the points $A, B, C, D,$ being fixed, E is also fixed, and

$\quad\therefore EK \times BK : DK \times KC$ in a constant ratio ;

$\quad\therefore PO \times PO' : PQ \times PQ'$ in a constant ratio.

Proposition XXXVI.

If from any point P on an ellipse, lines PR, PR', PS, PS', be drawn to the sides of an inscribed quadrilateral, making with them any constant angles, then the rectangles under the lines drawn to the opposite sides will be in a constant ratio.
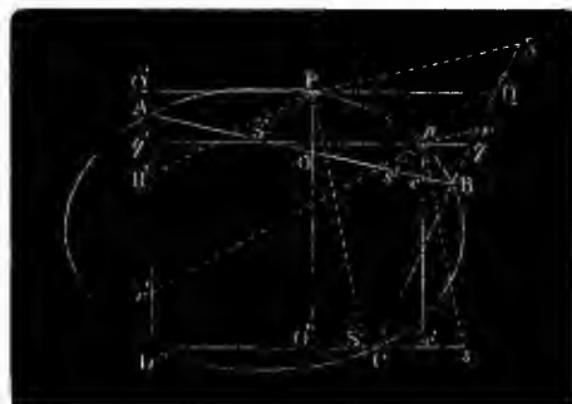

Fig. 49.

Take any other point p on the curve.

Through the points P and p draw QPQ' and $qpq' \parallel DC$, and POO', $poo' \parallel AD$, also pr, pr', ps, $ps' \parallel PR$, PR', PS, PS', respectively.

Then $PR : pr = PQ : pq$, (Similar $\triangle^s$.)

and $PR' : pr' = PQ' : pq'$; (Do.)

$$\therefore\ PR \times PR' : pr \times pr' = PQ \times PQ' : pq \times pq'.$$

Similarly it may be proved that

$$PS \times PS' : ps \times ps' = PO \times PO' : po \times po';$$

but $PQ \times PQ' : PO \times PO' = pq \times pq' : po \times po'$;

(Prop. xxxv.)

$$\therefore\ \ PR \times PR' : PS \times PS' = pr \times pr' : ps \times ps'.$$

Cor. 1.—The rectangle under the perpendiculars let fall from any point of an ellipse on two opposite sides of an inscribed quadrilateral is in a constant ratio to the rectangle under the perpendiculars let fall on the other two sides.

Cor. 2.—If the points A and B coincide, also the points C and D, then the sides AB and CD become tangents, and the sides BC and AD coincide, and become the chord of contact. Then the rectangle under the perpendiculars let fall from any point of an ellipse on two fixed tangents is in a constant ratio to the square of the perpendicular let fall on their chord of contact.

Cor. 3.—If we suppose AB, CD to intersect in X and AC, BD in Y; also that PR, PR', PS, PS', pr, pr', ps, ps' lie in the line XY; then, regarding $ACBD$ as the quadrilateral, the above proportion becomes

$$PX^2 : PY^2 = pX^2 : pY^2;$$

$$\therefore\ XY \text{ is cut harmonically by the curve.}$$

PROPOSITION XXXVII.

If two fixed tangents AD, DB, to an ellipse be cut by a diameter AB, parallel to their chord of contact, and by a third variable tangent EF, the rectangle under the segments of the two fixed tangents, intercepted between the diameter and the variable tangent, is constant.
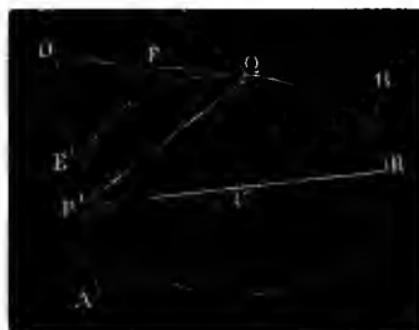
Fig. 50.

Join PC; produce it to meet the curve in R; join BR.

Since $\qquad\qquad CA = CB$; (Prop. xxxii.)

$\therefore$ the $\triangle^s ACP$ and BCR are equal; (4, I. Euclid.)

$\therefore BR$ is $=$ and $\parallel AP$;

$\therefore BR$ is a tangent.

Then $\qquad RB : PE = BF : FD$; (Cor. 2, Prop. xxvi.)

$\therefore AP : PE = BF : FD$;

$\therefore AP : AP + PE = BF : BF + FD$;

$\therefore AP : AE = BF : BD$;

$\therefore AE \times BF = BD \times AP$, and $\therefore$ constant.

LEMMA.

If on any right line we have segments AB, BC, &c., and on any other right line inclined to the former at any angle segments $A'B'$, $B'C'$, &c., such that $AB : A'B' = AC : B'C'$, &c. $= m : n$; then the locus of the middle points of AA', BB', CC', &c., will be a right line.

Fig. 51.

Measure off $C'O'$ and AO'', such that

$$CO : C'O' = AO'' : A'O = m : n.$$

Bisect OO'' in M and OO' in M'; join MM'.

This line will pass through the middle points of AA', BB', and CC'.

For $AB : A'B' = BC : B'C' = CO : C'O' = AO : A'O$;

$\therefore \frac{1}{2}(AB + BC + CO + AO'') : \frac{1}{2}(A'B' + B'C' + C'O' + A'O)$

$$= m : n;$$

or $\qquad\qquad OM : OM' = m : n.$

Again, $AB + AO'' : A'B' + A'O = m : n = MO'' : M'O$;

$\therefore AB + AO'' - MO'' : A'B' + A'O - M'O = m : n$;

or $\qquad\qquad MB : M'B' = m : n.$

Now consider the $\triangle OBB'$, the sides of which are cut by the transversal MM' ;

$$\therefore \frac{BF}{FB'} \cdot \frac{B'M'}{M'O} \cdot \frac{OM}{MB} = 1,$$

or
$$\frac{BF}{FB'} \cdot \frac{B'M'}{MB} \cdot \frac{OM}{M'O} = 1 ;$$

$$\therefore \frac{BF}{FB'} \cdot \frac{n}{m} \cdot \frac{m}{n} = 1 ;$$

$$\therefore BF = FB'.$$

In like manner it can be shown that AA' and CC' are bisected by MM'.

Proposition XXXVIII.

The right line joining the middle points of the diagonals of a quadrilateral circumscribing an ellipse will pass through the centre.
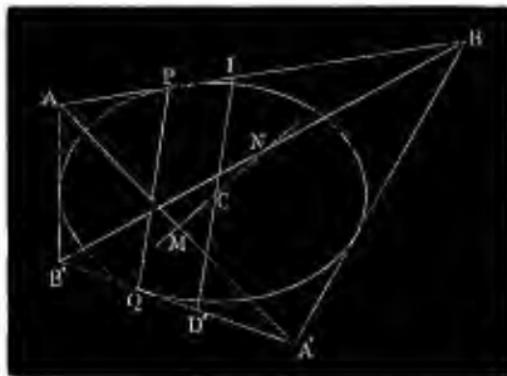
Fig. 52.

Let $ABA'B'$ be the quadrilateral. Through the centre C draw $DD' \parallel PQ$, the chord of contact of the opposite sides AB, $A'B'$.

Then $\qquad AD \times B'D' = BD \times A'D'$;

(Prop. xxxvii.)

$\therefore AD : A'D' :: BD : B'D'.$

Hence M, N, and C, the middle points of AA', BB', and DD', lie on a right line.

(Prop. xxxii., and Lemma, page 103.)

Proposition XXXIX.

If a quadrilateral be circumscribed to an ellipse, the diagonals will intersect on the chords of contact of the sides.
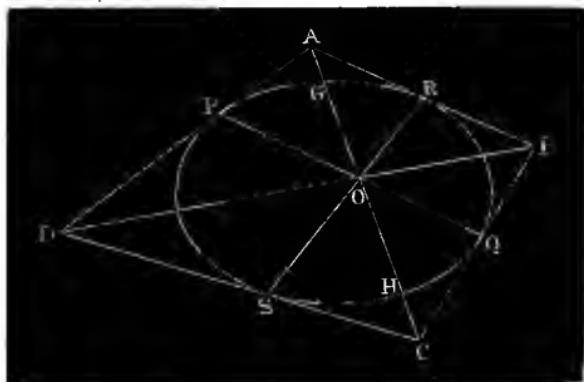
Fig. 53.

For AC will be cut by PQ, the chord of contact of the tangents DA, BC, so that

$$AO^2 : OC^2 = AH \times AG : CG \times CH ;$$

(Prop. xxxi.)

but it will also be cut by the chord of contact of the tangents AB, CD in the same ratio ;

$\therefore PQ$ and RS will intersect AC in the same point.

In like manner it may be shown that the diagonal DB will pass through the intersection of PQ and RS.

PROPOSITION XL.

The areas of the ellipse and auxiliary circle are in the ratio $CB : CA$.

Let the ellipse and auxiliary circle be divided by any number of lines drawn parallel to the axis minor.

Fig. 54.

Then, since

$$PM : P'M = QN : Q'N = CB : CA ;$$

$$(Cor.\ 3,\ \text{Prop. VIII.})$$

$\therefore$ quadrl. $PMNQ$: quadrl. $P'MNQ' = CB : CA$.

The same will be true of all the figures similarly described in the ellipse and auxiliary circle.

Hence the sum of all the quadrilateral figures inscribed in the ellipse : the sum of all the corresponding quadrilateral figures inscribed in the circle $= BC : CA$.

Now this will be true whatever be the number of the figures.

Let the number of the figures be increased, and the breadth of each indefinitely diminished. Then the sum of the areas of the figures inscribed in the ellipse will be

equal to the area of the ellipse, and the sum of the areas of those inscribed in the circle equal to the area of the circle;

$\therefore$ area of ellipse : area of circle = $CB : CA$.

PROPOSITION XLI.

The section of a right cone by a plane, which intersects all the generating lines on the same side of the vertex, will be an ellipse.
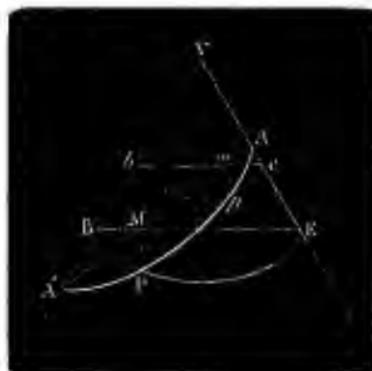
Fig. 55.

Let the plane BVE, drawn through the axis of the cone perpendicular to the plane of the *section*, coincide with the plane of the paper; then both the *section* APA' and the base BPE will be perpendicular to the plane of the paper; therefore the line MP, in which the section cuts the base, is perpendicular to the plane of the paper, and therefore perpendicular to BE, the diameter of the base; hence

$$BM \times ME = MP^2. \qquad \text{(35, III. Euclid.)}$$

If, now, any other plane bpe be drawn parallel to the base, meeting the section in pm, it can similarly be shown that mp is perpendicular to be; $\therefore$ $bm \times me = mp^2$.

But $BM : bm = A'M : A'm$; (Similar $\triangle^s$.)

also $ME : me = MA : mA$;

$\therefore BM \times ME : bm \times me = A'M \times MA : A'm \times mA$,

or $MP^2 : mp^2 = A'M \times MA : A'm \times mA$.

Hence the section APA' is an ellipse. (Prop. IX.)
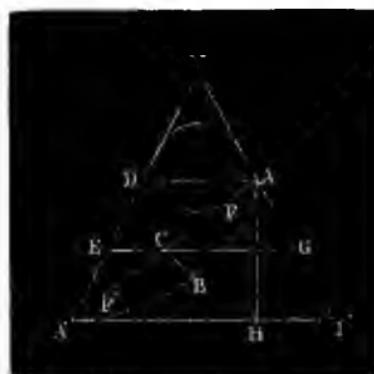
Fig. 56.

Cor. 1.—The axis minor is a mean proportional between the diameters of the frustrum of the cone which just includes the section.

Draw the section EBG parallel to the base, bisecting AA' in C.

Then $CE = \frac{1}{2} AD$ and $CG = \frac{1}{2} A'D'$; (Similar $\triangle^s$.)

$\therefore CE \times CG$ or $CB^2 = \frac{1}{4} AD \times A'D'$;

hence $4CB^2$ or the square of the axis minor $= AD \times A'D'$.

Cor. 2.—The Latus-rectum is a fourth proportional to AA', AD and $A'D'$.

Since ABA' is an ellipse,

$$L \times AA' = 4BC^2 \qquad (Cor. 1, \text{Prop. IX.})$$
$$= AD \times A'D';$$
$$\therefore AA' : AD = A'D' : L.$$

Cor. 3.—If CF and CF' be measured off $= AG$ or $\frac{1}{2} AD'$, then F and F' are the foci.

Draw $\quad AH \perp A'D'$.

Then $\quad AA'^2 - AD'^2 = A'H^2 - HD'^2$

$$= A'D' \times AD$$
$$\text{(Cor., 5, II. Euclid.)}$$

$$= 4BC^2; \qquad \text{(Cor. 1.)}$$

$$\therefore \quad CA^2 - AG^2 = BC^2;$$

hence $\quad CA^2 - BC^2 = AG^2 = CF^2 = CF'^2;$

$\therefore$ F and F' are the foci. $\qquad$ (Cor. 3, Prop. II.)

Cor. 4.—The spheres inscribed in a cone, to touch the plane of the section, will determine the foci.

For $\qquad AF = AC - CF$

$$= AC - AG \qquad \text{(Cor. 3.)}$$

$$= \tfrac{1}{2} (AA' - A'D)$$

$$= \tfrac{1}{2} (AA' + AV - VD - A'D)$$

$$= \tfrac{1}{2} (AA' + AV - VA');$$

$\therefore$ F is the point of contact of the circle inscribed in the triangle $AA'V$. (See Galbraith and Haughton's Manual of Euclid, Appendix, Book IV.)

In like manner it may be shown that F' is the point of contact of the circle exscribed to the triangle $AA'V$.

PROBLEMS ON THE ELLIPSE.

1. The axis major is the longest straight line that can be drawn in an Ellipse.

Join the two foci with the extremities of the chord; then twice the chord will be less than the sum of the four focal radii vectores, and therefore less than twice the axis major.

2. The minor axis is the least diameter of an Ellipse. Connect Problem 1 with *Cor.* 5, Prop. XVI.

3. Given the foci of an Ellipse, describe it so as to touch a given line.

See Prop. IV.

4. Find the locus of the point of intersection of any tangent to an Ellipse, with the line drawn from the focus making a constant angle with the tangent.

If the vertex of a triangle of a given species be fixed, while one base angle moves along a fixed circle, the locus of the other base angle will be a circle. See also Prop. VI.

5. The angle between the tangents at the extremities of a chord which passes through either focus is half the supplement of the angle which the chord subtends at the other focus.

Draw the normals at the extremities of the focal chord, and see *Cor.* 3, Prop. III.

6. Diameters parallel to supplemental chords are conjugate.

See Def., page 71, and *Cor.* 2, Prop. XVII.

7. If a rectangle be circumscribed to an Ellipse, its diagonals determine the equi-conjugate diameters.

8. Show that Prop. XXXIII. follows from Prop. XXXVI.

9. The line drawn parallel to the axis major through the intersection of normals at the extremities of a focal chord will bisect the chord.

See Prop. XIII.; also note properties of circles inscribed, exscribed, and circumscribed to a triangle.

10. When is the sum of two conjugate diameters a minimum; and when is it a maximum?

See *Cor.* 5, Prop. XVI.; *Cor.*, Prop. XXI., and Prop. XXIX.

CHAPTER III.

THE HYPERBOLA.

DEFINITIONS.

A HYPERBOLA is the curve traced out by a point which moves in such a way that its distance from a fixed point is to its distance from a fixed right line in a constant ratio $\epsilon : 1$ (ϵ being greater than unity).

The fixed point is called the *Focus*, and the fixed right line the *Directrix*.

For Definitions of *Axis, Vertex, Centre, Diameter, Tangent, Normal, Chord, Ordinate, Abscissa, Latus rectum*, &c., see Definitions, Chapters I. and II.

Conjugate Diameters may be defined, as on page 72. *Supplemental Chords, Transverse Axis, Conjugate Axis*, as on page 71.

Two hyperbolas are said to be *conjugate* when the transverse axis of each is the conjugate axis of the other.

If the axes be equal the hyperbola is said to be *Rectangular* or *Equilateral*.

Tangents to the curve whose points of contact are at infinity are called *Asymptotes*.

PROPOSITION I.

The focus, directrix, and eccentricity of a hyperbola being given, to determine any number of points on the curve.
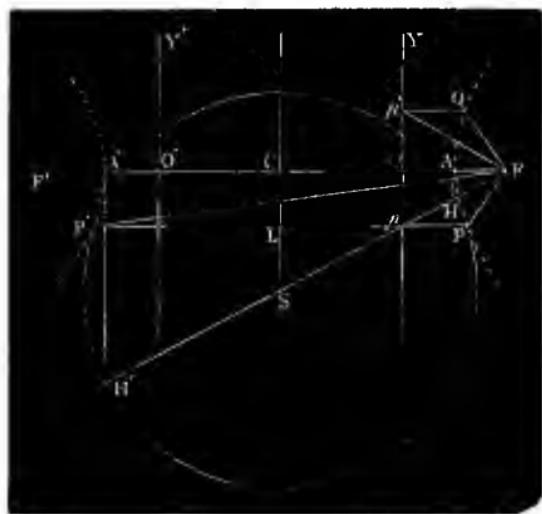
Fig. 1.

Let F be the focus, and Oy the directrix.

Draw $FO \perp$ the directrix. Divide FO internally at A, and externally at A', so that $FA : AO = FA' : A'O$ in the given ratio $\varepsilon : 1$; then A and A' are the vertices of the curve, and AA' the axis.

On the directrix take any point p; join FP, and produce it; draw AH, $A'H'$ perpendiculars to AA', meeting Fp in H, H'. On HH' as diameter describe a circle; through p draw $PpP' \parallel$ to the axis, cutting the circle in P, P'; join PF, $P'F$.

Then, since AH is $\parallel Op$; $\therefore HF : Hp = AF : AO$.

Also, since $A'H'$ is $\parallel Op$; $\therefore H'F : H'p = A'F : A'O$;

$$\therefore H'F : H'p = HF : Hp.$$

Hence $\quad PF : Pp = FH : Hp \quad$ (Lemma, page 47.)

$$= FA : AO = \varepsilon : 1. \quad (2, \text{VI. Euclid.})$$

Also $\quad P'F : P'p = FH : Hp \quad$ (Lemma, page 47.)

$$= FA : AO = \varepsilon : 1. \quad (2, \text{VI. Euclid.})$$

$\therefore$ P and P' are points on the curve.

In like manner, by taking other points on the directrix, any number of points on the curve may be determined.

Cor. 1.—If Op' be taken equal to Op, and another point Q found in a similar manner to P, it is obvious that Qp' will be equal to Pp. Therefore, corresponding to any point P on the curve, at a perpendicular distance from the axis $= Op$, there is another point Q on the other side of the axis, at a distance from it $= Op$, and also at the same distance as P from the directrix; hence the curve is symmetrical with regard to the axis.

Cor. 2.—Bisect AA' in C; through C draw $CLS \perp AA'$, and $\therefore \perp PP'$, meeting HH' in S, and PP' in L.

Then, since AH, CS and AH' are parallel, $\therefore SH = SH'$;

$\qquad \therefore$ S is the centre of the circle $H'P'HP$;

$\qquad \therefore LP = LP'.$ $\qquad\qquad\qquad$ (3, III. Euclid.)

Therefore, corresponding to any point P on the curve, there is another point P' on the other side of CL, situated in precisely the same manner with regard to it as P; hence the curve is symmetrical also with regard to CL.

If therefore CO' be measured off $= CO$, and $CF' = CF$, and $O'Y'$ be drawn $\perp O'C$, the curve could be equally well described with F' as focus, and $O'Y'$ as directrix.

Cor. 3.—Since $PF : Pp = AF : AO = A'F : A'O$;

$$\therefore PF : Pp = A'F - AF : A'O - AO$$

$$= 2CA : 2CO ;$$

$$\therefore PF : Pp = CA : CO.$$

I

That is, the distance of any point on the curve from the focus is to its distance from the directrix as $CA : CO$.

Cor. 4.—Since $A'F : A'O = AF : AO$;

$$\therefore A'F + AF : A'F - AF = A'O + AO : A'O - AO ;$$

or $$2CF : 2CA = 2CA : 2CO ;$$

$$\therefore CF \times CO = CA^2.$$

Cor. 5.—From the symmetry of the curve it is evident that any line drawn through C to meet the curve will be bisected at that point; hence C is the centre of the curve.

PROPOSITION II.

The difference of the distances of the foci from any point on a hyperbola is equal to the transverse axis.

Fig. 2.

$$PF' : Pp' = CA : CO; \quad (\text{\textit{Cor.} 3, Prop. I.})$$

also $$PF : Pp = CA : CO ;$$

$$\therefore PF' - PF : Pp' - Pp = CA : CO ;$$

but $$Pp' - Pp = OO' = 2CO ;$$

$$\therefore PF' - PF = 2CA = AA'.$$

Cor. 1.—By the help of the preceding theorem we can construct a hyperbola mechanically.

If two of the extremities of two threads, the difference of whose lengths is equal to AA', be fastened to the two fixed points A, A', and the other extremities, having been passed through the eye of a needle, be joined, it is obvious that the eye of the needle, moved about so as to keep the threads always stretched, will describe a hyberbola whose foci are F and F'.

Cor. 2.—The difference of the distances of any point from the focii of a hyperbola is greater or less than the transverse axis, according as the point is on the concave or convex side of the curve.
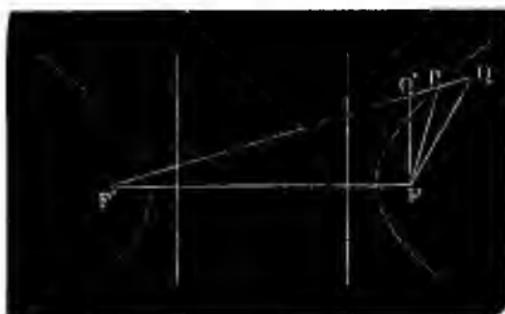
Fig. 3.

First let Q be any point on the concave side of the curve.

Join QF, QF', and let QF' meet the curve in P; join PF.

Then $\qquad QF' = QP + PF'$,

and $\qquad QF < QP + PF$; $\qquad$ (20, I. Euclid.)

$\qquad \therefore QF' - QF > PF' - PF$;

but $\qquad PF' - PF = AA'$; $\therefore QF' - QF > AA'$.

I 2

Next, let Q' be on the convex side.

Join $Q'F$, $Q'F'$; produce $Q'F'$ to meet the curve in P; join PF.

Then $\qquad F'Q' = F'P - PQ'$,

and $\qquad Q'F > PF - PQ'$; (*Cor.*, 20, I. Euclid.)

$$\therefore \; Q'F' - Q'F < PF' - PF;$$

but $\qquad PF' - PF = AA'$; $\therefore \; Q'F' - Q'F < AA'$.

Cor. 3.—Conversely, a point will be on the concave or convex side of a hyperbola, according as the difference of its distances from the foci is greater or less than the transverse axis.

Proposition III.

1°. The line which bisects the angle formed by drawing lines from any point P on a hyperbola to the foci lies altogether without the curve.

2°. Any other line drawn through the point P will cut the curve.

Fig. 4.

1°. Let PH bisect the $\angle$ between PF and PF'.

Cut off $PR = PF$; join F, F' and R with *any* point Q on the line PH.

Then the $\triangle^s$ QFR and QRP are equal; (4, I. Euclid.)

$$\therefore QR = QF;$$

but $\qquad F'Q - QR < F'R;$ (*Cor.*, 20, I. Euclid.)

$$\therefore F'Q - QF < F'R$$
$$< F'P - PR$$
$$< F'P - PF \qquad \text{(Const.)}$$
$$< AA'; \qquad \text{(Prop. II.)}$$

$\therefore$ Q is a point without the curve. (*Cor.* 3, Prop. II.)

$2°$. Let PK be any line other than the bisector of the $\angle FPF'$.

Fig. 5.

Draw PL, making the $\angle KPL = \angle KPF$; cut off $PL = PF$.

Join $F'L$; produce $F'L$ to meet PK in Q; join QF.

Then the $\triangle^s$ FPQ and LPQ are equal; (4, I. Euclid.)

$$\therefore QF = QL;$$

but $\qquad F'L > F'P - PL$

(*Cor.*, 20, I. Euclid.)

$$> F'P - PF$$
$$> AA'; \qquad \text{(Prop. II.)}$$
$$\therefore F'Q - QL \text{ or } F'Q - QF > AA';$$

$\therefore$ Q is a point on the concave side of the curve.

(*Cor.* 3, Prop. II.)

Cor. 1.—Hence the line which bisects the angle between lines drawn from any point on a hyperbola to the foci is a tangent.

Cor. 2.—The tangent at the vertex is perpendicular to the axis.

Cor. 3.—The normal to a hyperbola at any point bisects the external angle between the focal radii vectores drawn to the point.

Proposition IV.

The locus of the foot of the perpendicular let fall from either focus on any tangent to a hyperbola is a circle described on the transverse axis as diameter.
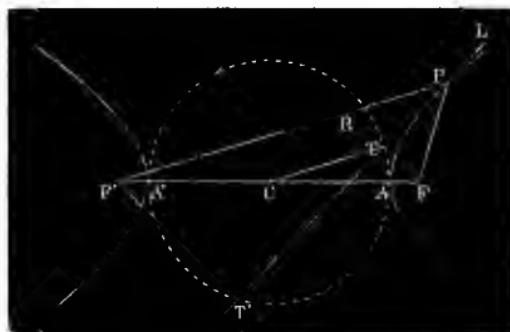
Fig. 6.

Let PT be any tangent, and FT a perpendicular let fall from the focus.

Produce FT to meet $F'P$ in R; join TC.

Then the $\triangle^s FTP$ and RTP are equal; (26, I. Euclid.)

$$\therefore FT = TR.$$

Now, in $\triangle F'FR$, $F'F$ is bisected in C, and FR in T;

$$\therefore CT = \tfrac{1}{2} FR = \tfrac{1}{2} (F'P - PR)$$

$$= \tfrac{1}{2} (F'P - PF)$$

$$= \tfrac{1}{2} AA' \qquad \text{(Prop. ii.)}$$

$$= CA.$$

Hence the locus of T is a circle described with the centre C and radius CA.

Cor. 1.—If the vertex of a right angle FTL move along a fixed circle, while one leg passes through a fixed point F without that circle, the other leg will always touch a hyperbola.

Draw the diameter of the circle passing through F; cut off $CF' = CF$; produce FT till $TR = TF$; join $F'R$, and produce it to meet TL in P; then P is the point at which TL will touch the hyperbola.

Join FP, CT.

Then the $\triangle^s FTP$ and RTP are equal;   (4, I. Euclid.)

$$\therefore FP = RP.$$

Also, since $FT = TR$ and $FC = CF'$;

$$\therefore CT \parallel F'R \text{ and } = \tfrac{1}{2} F'R ;$$

$$\therefore PF' - PF = PF' - PR$$

$$= F'R$$

$$= 2CT$$

$$= 2CA ;$$

hence P is a point on the hyperbola of which F and F' are the foci, and AA' the transverse axis.

Cor. 2.—If *FT* become a tangent to the circle, then *CT* will be $\perp$ *FT*, and $\therefore$ in directum with *TL*; and $\therefore$ *F'R*, which is $\parallel$ *CT*, will meet *TL* at infinity; hence *CL* is an asymptote to the curve.
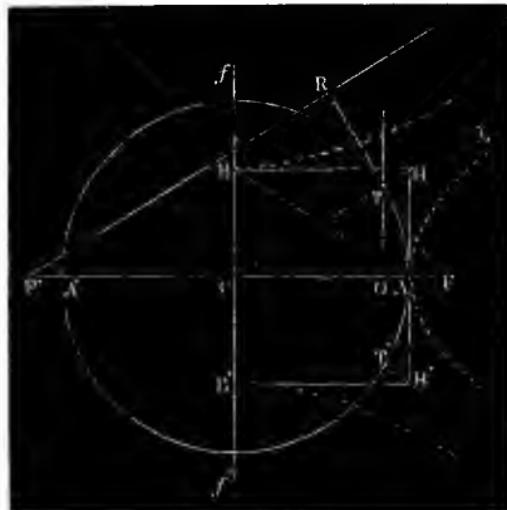
Fig. 7.

If *FT'* be the other tangent to the circle drawn from *F*, then *CT'* will also be an asymptote.

Cor. 3.—If *TO* be drawn $\perp$ the axis, then *TO* is the directrix.

For $$CF \times CO = CA^2; \quad \text{(8, VI. Euclid.)}$$

$\therefore$ *O* is a point on the directrix. $\quad \cdot$ (*Cor.* 4, Prop. I.)

Cor. 4.—If *AH* be drawn $\perp$ *CF*, meeting *CL* in *H*, and *HB* be drawn $\parallel$ *CF*, meeting *CB* drawn $\perp$ *CF* in *B*, then *CB* is the *semi-conjugate axis*.

For the $\triangle^s$ *CFT* and *CHA* are equal; (26, I. Euclid.)

$$\therefore TF = AH = CB;$$

$$\therefore CB^2 = FT^2 = FA \times FA'.$$
(36, III. Euclid.)

Hence the point B, together with the corresponding point B', obviously correspond to the extremities of the axis minor in the Ellipse.   (See *Cor.* 3, Prop. II., page 51.)

Cor. 5. $CF^2 = CA^2 + CB^2$.

For $CF^2 = CH^2 = CA^2 + AH^2 = CA^2 + CB^2$.

Cor. 6.—If we cut off Cf and Cf' each $= CF$ or CH, and consider f, f' the foci, and BB' the transverse axis of a hyperbola, then AA' will be its conjugate axis.

For the square of its semi-conjugate axis

$$= Cf^2 - CB^2 \qquad (Cor.\ 5.)$$

$$= CH^2 - AH^2$$

$$= CA^2.$$

The hyperbola described with BB' as the transverse axis, and f, f' as foci, is called the hyperbola *conjugate* to the original hyperbola.

Cor. 7.—The lines joining the extremities of the axes are bisected by one asymptote and parallel to the other.

For $CAHB$ is a rectangle ; $\therefore AB$ is bisected by CH.

Also, $CB = AH = AH'$ and $CB \parallel AH'$;

$$\therefore AB \parallel CH'.$$
$$\text{(33, I. Euclid.)}$$

Cor. 8.—The perpendicular from the focus on either asymptote is equal to the semi-conjugate axis.

For $\triangle\ CAH = \triangle\ CTF;$ (26, I. Euclid.)

$$\therefore FT = AH = CB.$$

NOTE.—When *the axis* only is mentioned it is always understood to be the transverse axis.

PROPOSITION V.

The distance of any point P on the curve from the focus F is equal to the length of the line Pq drawn from the point parallel to an asymptote to meet the directrix.
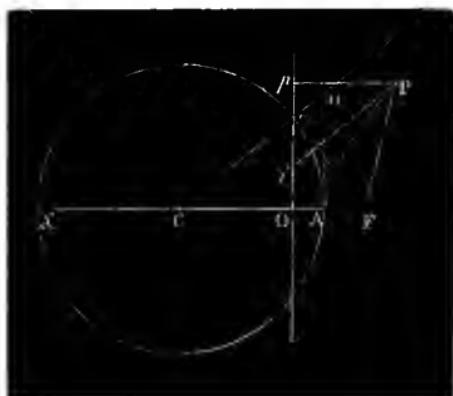
Fig. 8.

On the transverse axis as diameter describe a circle.
Draw the tangent FT; join CT, and draw $TO \perp CF$.
Then OT is the directrix, and CT an asymptote.

$$\text{(Cors. 2 \& 3, Prop. iv.)}$$

Also $Pq : Pp = CT : CO$ (Similar $\triangle^s$.)
$$= CA : CO$$
$$= PF : Pp. \quad \text{(Cor. 3, Prop. i.)}$$
Hence, $PF = Pq.$

Cor.—Hence, being given the eccentricity of a hyperbola, we can find the angle between the asymptotes.

For $CT : CO = Pq : Pp$ (Similar $\triangle^s$.)
$$= PF : Pp$$
$$= \epsilon : 1.$$

Now in the $\triangle CTO$ we know $\angle$ at $O = 90°$; also the ratio of the sides $(CT : CO)$; hence the $\angle TCO$, or half the $\angle$ between the asymptotes, is known.

Proposition VI.

The rectangle under the focal perpendiculars on any tangent to a hyperbola is equal to the square of the semi-conjugate axis.
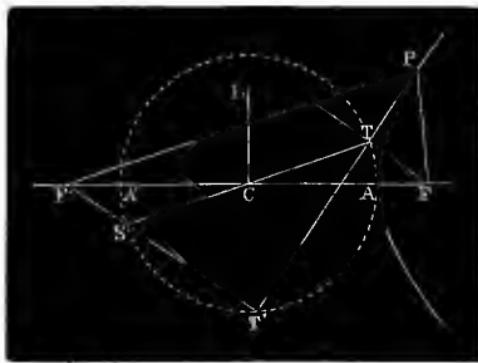
Fig. 9.

Let FT, $F'T'$, be perpendiculars from the foci on the tangent at any point P.

Join TC, and produce it to meet $F'T'$ in S.

Then the $\triangle^s FCT$ and $F'CS$ are equal;

$$\text{(26, I. Euclid.)}$$

$$\therefore\ F'S = FT, \text{ and } CS = CT = CA;\quad \text{(Prop. IV.)}$$

$$\therefore\ S \text{ is a point on the auxiliary circle};$$

$$\therefore\ A'F' \times F'A = F'T' \times F'S \quad \text{(35, III. Euclid.)}$$

$$= F'T' \times FT;$$

$$\therefore\ F'C^2 - CA'^2 = F'T' \times FT,\quad \text{(5, II. Euclid.)}$$

or $\qquad\qquad BC^2 = F'T' \times FT.$ $\quad$ (*Cor.* 5, Prop. IV.)

PROPOSITION VII.

The locus of the intersection of tangents to a hyperbola which cut at right angles is a circle.
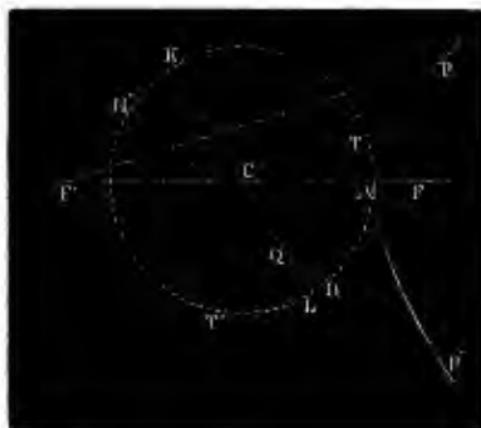
Fig. 10.

Describe the auxiliary circle cutting the tangent QP in T and T'', and the tangent QP' in H and H'. Join FH, FT, $F'H'$, $F'T'$, QC; produce QC to meet the auxiliary circle in K and L.

Then FT, $F'T''$, are $\perp QP$, and $\therefore \parallel QP'$;

(Prop. IV.)

and FH, $F'H' \perp QP'$; (Prop. IV.)

hence $F'T'' = QH'$, and $FT = QH$. (34, I. Euclid.)

Now $CQ^2 = CL^2 - KQ \times QL$ (5, II. Euclid.)

$= CL^2 - QH' \times QH$ (35, III. Euclid.)

$= CA^2 - FT \times F'T'$

$= CA^2 - BC^2$. (Prop. v.)

Hence the locus of Q is a circle described with the centre C, and is called *the director circle* of the hyperbola.

PROPOSITION VIII.

If PN be a normal at any point P on a hyperbola, and PM an ordinate to the transverse axis; then

$$CA^2 : CF^2 :: CM : CN.$$
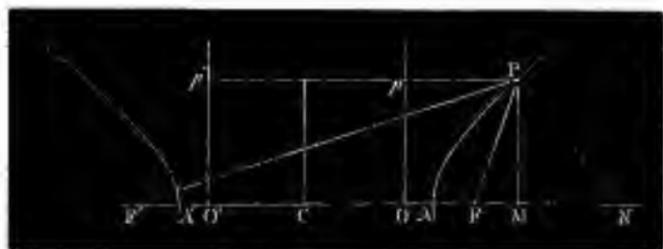
Fig. 11.

Join PF, PF'; draw Ppp' ∥ the axis, meeting the directrices in p, p'.

Since PN bisects the external $\angle$ between PF and PF';

(*Cor.* 3, Prop. III.)

$$\therefore F'N : NF = PF' : PF$$

(Prop. B., VI. Euclid.)

$$= Pp' : Pp.$$

Comp. $F'N + NF : F'N - NF = Pp' + Pp : Pp' - Pp$;

$$\therefore 2CN : 2CF = 2CM : 2CO.$$

Alt. $CN : CM = CF : CO$

$$= CF^2 : CF \times CO$$

$$= CF^2 : CA^2.$$

(*Cor.* 4, Prop. I.)

Cor. $CM : MN = CA^2 : CB^2.$

For $CM : CN - CM = CA^2 : CF^2 - CA^2$;

or $CM : MN = CA^2 : CB^2.$

(*Cor.* 4, Prop. IV.)

PROPOSITION IX.

If a tangent at any point P of a hyperbola be produced to meet the axis in T, and the ordinate PM be drawn; then CA will be a mean proportional between CM and CT.
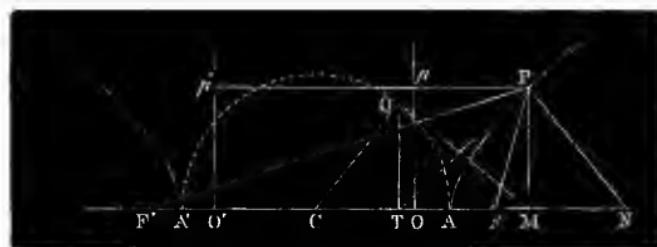
Fig. 12.

Through P draw Ppp' ∥ to the axis; join PF, PF'.

Then, since PT is the bisector of the $\angle F'PF$;

(*Cor.* 1, Prop. III.)

$$\therefore F'T : TF = F'P : PF \quad (3, \text{VI. Euclid.})$$

$$= Pp' : Pp.$$

Comp. and div.

$$F'T + TF : F'T - TF = Pp' + Pp : Pp' - Pp ;$$

or
$$2CF : 2CT = 2CM : 2CO ;$$

$$\therefore CT \times CM = CF \times CO$$

$$= CA^2. \quad (\textit{Cor.} 4, \text{Prop. I.})$$

Cor. 1.—If the auxiliary circle be described, and TQ drawn $\perp$ the axis; then QM is a tangent to the circle. Join CQ.

Since
$$CT : CA = CA : CM;$$

$$\therefore CT : CQ = CQ : CM;$$

$$\therefore \text{ the } \triangle^s CQT \text{ and } CMQ \text{ are similar};$$

(6, VI. Euclid.)

$$\therefore \angle CQM = \angle CTQ ; \therefore QM \text{ is a tangent.}$$

Cor. 2.—If from the foot of any ordinate of a hyperbola a tangent be drawn to the auxiliary circle, this tangent is to the ordinate in a constant ratio.

Draw the normal PN.

Then, since the $\triangle TPN$ is right-angled, and $PM \perp TN$;

$$\therefore\ TM \times MN = PM^2 \qquad \text{(8, VI. Euclid.)}$$

Also, since $\triangle CQM$ is right-angled, and $QT \perp CM$;

$$\therefore\ CM \times TM = QM^2$$

Hence $\qquad QM^2 : PM^2 = CM : MN$

$$= CA^2 : CB^2;$$

$$(\textit{Cor.}, \text{Prop. VIII.})$$

$$\therefore\ QM : PM = CA : CB.$$

Cor. 3.—If QA be drawn and produced to meet PM produced in K, then

$$MK = MQ, \text{ and } PM : MK = CB : CA.$$
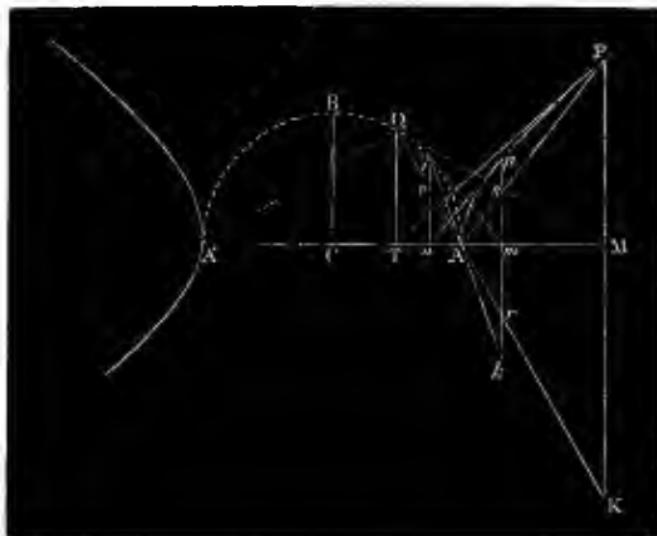
Fig. 13.

For $\quad\quad\quad \angle MQA = \angle QA'A \quad\quad$ (32, III. Euclid.)

$\quad\quad\quad\quad\quad\quad = \angle AQT \quad\quad\quad$ (13, VI. Euclid.)

$\quad\quad\quad\quad\quad\quad = \angle AKM;$

$\quad\quad\quad \therefore MK = MQ;$

Also $\quad\quad\quad PM : QM = CB : CA ;$

$\quad\quad\quad \therefore PM : MK = CB : CA.$

Cor. 4.—The curve is concave towards the transverse axis.

(See Fig. 13.)

Draw any line perpendicular to the axis between the vertex A and the point P; let it meet the curve in p, AP in S, the axis in m, and QA produced in r. From m draw the tangent mq to the auxiliary circle; draw qn $\perp$ to the axis; join np; produce qA to meet pr produced in k.

Then $\quad\quad PM : MK = CB : CA = pm : mk;$ (*Cor.* 2.)

but $\quad\quad\quad PM : MK = sm : rm;$ $\quad\quad$ (Similar Δ^s.)

$\quad\quad\quad \therefore pm : mk = sm : rm.$

Alt. $\quad\quad\quad pm : sm = mk : rm$

$\quad\quad\quad\quad\quad\quad = qn : vn;$

but qn is $> vn$ by property of the circle;

$\quad\quad\quad \therefore pm$ is $> sm.$

That is, if a perpendicular be drawn to the transverse axis produced, between the vertex A and any point P on the branch of the curve of which A is the vertex, the segment intercepted on it between the curve and the transverse axis is greater than the segment intercepted on it between the line AP and the transverse axis; and, therefore, the curve is concave towards the transverse axis.

PROPOSITION X.

The rectangle under the segments of the axis made by any ordinate is to the square of the ordinate in a constant ratio.

See Fig. 12.

For $\quad PM^2 : QM^2 = CB^2 : CA^2$; $\quad$ (*Cor.* 2, Prop. IX.)

but $\quad\quad\quad QM^2 = MA \times MA'$; $\quad$ (36, III. Euclid.)

$$\therefore PM^2 : MA \times MA' = CB^2 : CA^2.$$

Cor. 1.—The Latus rectum is a third proportional to the transverse and conjugate axes.

$$\left(\frac{L}{2}\right)^2 : AF \times FA' = CB^2 : CA^2;$$

$$\therefore \left(\frac{L}{2}\right)^2 : CB^2 = CB^2 : CA^2.$$

Hence $\quad\quad CA : CB = CB : \dfrac{L}{2}.$ $\quad$ (*Cor.* 4, Prop. IV.)

Cor. 2. $\quad\quad\quad CB^2 = CM' \times CT'.$
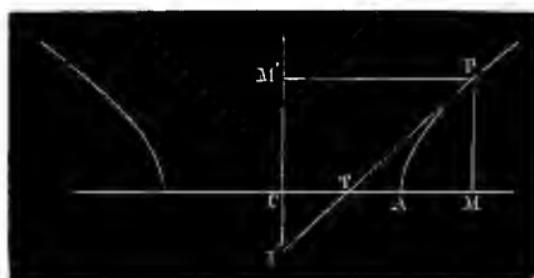
Fig. 14.

For $\quad\quad CM^2 : CA^2 = CM : CT$; $\quad$ (Prop. IX.)

$$\therefore CM^2 - CA^2 : CA^2 = CM - CT : CT.$$

$$PM^2 : CB^2 = TM : CT \quad\quad \text{(Prop. x.)}$$

$$= PM : CT \quad\quad \text{(Similar } \triangle\text{'s.)}$$

$$= PM^2 : PM \times CT';$$

$$\therefore CB^2 = CM' \times CT'.$$

K

PROPOSITION XI.

The locus of the intersection of any tangent to a hyperbola, with the line drawn through the focus perpendicular to the radius vector drawn from the focus to the point of contact of the tangent, is the directrix.

Fig. 15.

Let PT be any tangent, and $FT \perp FP$.

Draw $TH \perp PF'$, and $TO \perp$ the axis.

The $\triangle^s PFT$ and PHT are equal.   (26, I. Euclid.)

$\therefore FT = TH$, and $PH = PF$;

$\therefore F'H = F'P - PH = F'P - PF = AA'$.

(Prop. II.)

Now $\qquad F'H^2 = F'T^2 - TH^2 \qquad$ (47, I. Euclid.)

$\qquad\qquad = F'T^2 - TF^2$

$\qquad\qquad = F'O^2 - OF^2$;

$\therefore AA'^2 = 4CF \times CO$;   (*Cor.* 5, II. Euclid.)

$\therefore CA^2 = CF \times CO$.

Hence OT is the directrix.   (*Cor.* 4, Prop. I.)

Cor.—Conversely, if from any point on the directrix a tangent be drawn to a hyperbola, the line joining that point to the focus is perpendicular to the radius vector drawn from the focus to the point of contact of the tangent.

Proposition XII.

If any chord PP' of a hyperbola cut the directrix in D, and if F be the focus corresponding to the directrix, on which D is situated; then FD is the external bisector of the $\angle PFP'$.

Fig. 16.

Draw Pp, $P'p'$ $\perp$ the directrix; produce $P'F$.

Then $\qquad PF : Pp = P'F : P'p'$.

Alt. $\qquad PF : P'F = Pp : P'p'$

$\qquad\qquad\qquad = PD : P'D;$ $\qquad$ (Similar $\triangle^s$.)

$\therefore$ FD bisects the $\angle PFQ$. $\qquad$ (Prop. B., VI. Euclid.)

Cor. 1.—Hence, being given one focus F, and three points P, P', P'', on a hyperbola, we can find the directrix and axes.

For, draw FD bisecting the angle between PF and $P'F$ produced; the point where this line meets $P'P$ produced will be one point on the directrix. Similarly, by bisecting the angle between PF and $P''F$ produced, another point on the directrix may be found; and hence the directrix and axes.

If a tangent be defined as the line joining two indefinitely near points on the curve, it will follow immediately from the proposition that the right line drawn from the focus to the point of intersection of any tangent with the directrix is perpendicular to the radius vector drawn from the focus to the point of contact of the tangent; also, that the tangent bisects the angle between the radii vectores drawn from the foci to the point of contact.

For when P is indefinitely near to P', PD becomes a tangent at the point P, and the $\angle PFP'$ is indefinitely small (see Fig. 16);

$$\therefore \angle PFQ = 180°,$$

but $$\angle PFD = \tfrac{1}{2} \angle PFQ;$$

$$\therefore \angle PFD = 90°.$$

Also (see Fig. 17),
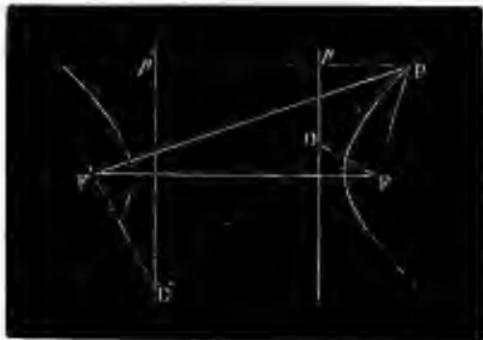
Fig. 17.

$$PF : PF' = Pp : Pp'$$

$$= PD : PD', \qquad \text{(Similar } \triangle^s.)$$

and $$\angle PFD = PF'D', \text{ each being } 90°;$$

$$\therefore \triangle PFD \text{ is similar to } \triangle PF'D';$$

hence $$\angle FPD = \angle F'PD'.$$

PROPOSITION XIII.

If two fixed points, P, P', on a hyperbola be joined with a third variable point O, the segment pp' intercepted on either directrix by the produced chords subtend a constant angle at the focus corresponding to that directrix.
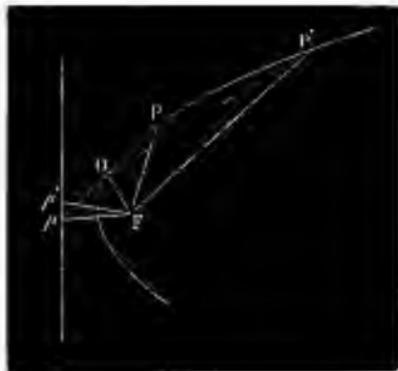
Fig. 18.

Since Fp is the external bisector of the $\angle PFO$;

(Prop. XII.)

$$\therefore \angle OFp + \tfrac{1}{2} \angle OFP = 90°.$$

Also Fp' is the external bisector of the $\angle OFP'$;

(Prop. XII.)

$$\therefore \angle OFp' + \tfrac{1}{2} \angle OFP' = 90°;$$
$$\therefore \angle OFp + \tfrac{1}{2} \angle OFP = \angle OFp' + \tfrac{1}{2} \angle OFP';$$
$$\therefore \angle OFp - \angle OFp' = \tfrac{1}{2} (\angle OFP' - \angle OFP)$$

or $\qquad \angle pFp' = \tfrac{1}{2} \angle PFP'$, and $\therefore$ constant.

Cor.—The anharmonic ratio of the pencil formed by joining four fixed points on a hyperbola to any fifth variable point is constant.

For $\qquad O \cdot PP'P''P''' = O \cdot pp'p''p''' = F \cdot pp'p''p'''.$

Proposition XIV.

The line joining the focus to the intersection of two tangents to a hyperbola bisects the angle which the points of contact subtend at the focus.
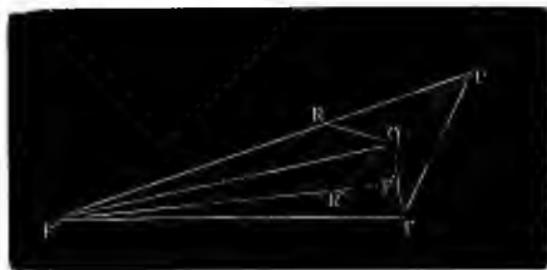
Fig. 19.

Let QP, QP' be tangents; join PF, PF'', $P'F$, $P'F'$. Cut off $PR = PF$, and $P'R' = P'F$; join Q with R, R', F, F'.

Then the $\triangle^s$ QFP and QRP are equal;

(4, I. Euclid.)

$$\therefore \ QR = QF.$$

Again, the $\triangle^s$ QFP' and $QR'P'$ are equal;

(4, I. Euclid.)

$$\therefore \ QR' = QF;$$

hence $\qquad QR = QR'$;

but $\qquad F'R = F'P - PR$

$$= F'P - PF$$
$$= P'F' - P'F \qquad \text{(Prop. xi.)}$$
$$= P'F' - P'R'$$
$$= F'R'.$$

Hence the $\triangle^s$ $RF'Q$ and $R'F'Q$ are equal;

(8, I. Euclid.)

$$\therefore \ \angle RF'Q = \angle R'F'Q.$$

Cor.—It is obvious from the above demonstration, also, that

$$\angle PFQ = \angle PRQ = \angle QR'P'$$
$$= \angle QFP'.$$

PROPOSITION XV

The angle subtended at either focus by the segment intercepted on a variable tangent to a hyperbola by two fixed tangents is constant.

Fig. 20.

Let QP, QP' be the two fixed tangents, and RS the variable tangent.

Join FP, FS, FT, FR, FP'.

Then $\angle TFS = \frac{1}{2} \angle TFP$; also $\angle TFR = \frac{1}{2} \angle TFP'$;

(Prop. XIII.)

$\therefore \angle TFS + \angle TFR = \frac{1}{2} \angle TFP + \frac{1}{2} \angle TFP'$,

or $\qquad \angle RFS = $ a constant.

Cor.—The anharmonic ratio of the four points in which a variable tangent to a hyperbola is cut by four fixed tangents is constant.

For the segments intercepted on the variable tangent subtend constant angles at the focus.

PROPOSITION XVI.

If a tangent at the extremity of any diameter meet the axis in T, the area of the triangle CPT thus formed will be equal to the area of the triangle CAH, formed by drawing a tangent at the vertex.

Fig. 21.

Draw the ordinate PM; join PA, TH.

Then $\qquad TC : CA = CA : CM \qquad$ (Prop. IX.)

$\qquad\qquad\qquad = CH : CP ; \qquad$ (Parallel lines.)

$\qquad\qquad \therefore\ PA$ is $\parallel HT; \qquad$ (2, VI. Euclid.)

$\qquad\qquad \therefore\ \triangle PHT = \triangle HTA ; \qquad$ (37, I. Euclid.)

hence $\qquad\qquad \triangle CPT = \triangle CAH.$

Cor. $\qquad\qquad \triangle PMT = \text{trap. } PMAH.$

If we take the equal $\triangle^s\ CPT$ and CAH, in succession from the $\triangle CPM$;

then $\qquad\qquad \triangle PMT = \text{trap. } PMAH.$

PROPOSITION XVII.

If an ordinate to the axis drawn from any point P on a hyperbola be produced to meet the asymptotes in D and D', then $PD \times PD'$ = square of semi-conjugate axis.
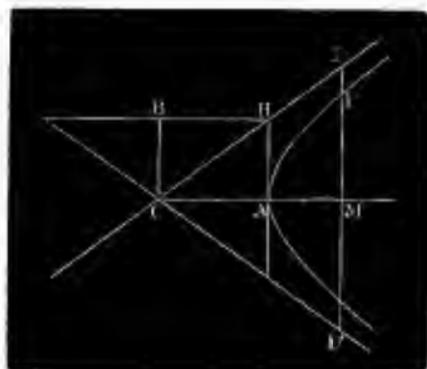
Fig. 22.

For $\quad MD^2 : MC^2 = AH^2$ or $BC^2 : AC^2$;

(Similar $\triangle^s$.)

but $\quad PM^2 : MC^2 - CA^2 = BC^2 : CA^2$; (Prop. x.)

$\therefore \ MD^2 - PM^2 : CA^2 = BC^2 : CA^2$;

hence $\quad MD^2 - PM^2 = BC^2$,

or $\quad PD \times PD' = BC^2$. (5, II. Euclid.)

NOTE.—In the figures connected with some of the following Propositions, in order to avoid confusion, occasionally lines are not actually drawn : when a letter is omitted, reference to any preceding figure, where the same letter has been used, will at once indicate the point.

PROPOSITION XVIII.

The segments PQ, $P'Q'$ intercepted on *any* line between the curve and its asymptotes are equal.

Fig. 23.

Through P and P' draw $\perp^s$ to the axis.

Then $\qquad QP : PR = QP' : P'S$, $\qquad$ (Similar Δ^s.)

and $\qquad PQ' : Pr = Q'P' : P'S'$;

$\therefore QP \times PQ' : PR \times Pr = QP' \times Q'P' : P'S \times P'S'$;

but $\qquad PR \times Pr = BC^2 = P'S \times P'S'$ $\qquad$ (Prop. XVII.)

$\therefore QP \times PQ' = QP' \times Q'P',$

or $QP \times PP' + QP \times Q'P' = Q'P' \times PP' + Q'P' \times QP$;

$\qquad\qquad\qquad\qquad\qquad\qquad$ (1, II. Euclid.)

$\therefore QP \times PP' = Q'P' \times PP',$

or $\qquad\qquad\qquad QP = Q'P'.$

Cor. 1.—If a line drawn from C through the middle point of QQ' meet the curve in D, then the line drawn through D parallel to QQ' will be a tangent.

For, suppose it meet the curve again in D', and let it meet the asymptotes in E and E'.

Then, by this Proposition, $DE = D'E'$.

But in the $\triangle\ QCQ'$, since QQ' is bisected in G, and $EE' \parallel QQ$;

$$\therefore\ DE = DE'.$$

Hence $DE' = D'E'$, which is impossible; $\therefore$ &c.

Cor. 2.—Any line PP' is an ordinate to the diameter passing through its middle point, and the portion of any tangent intercepted between the asymptotes is bisected at its point of contact.

Cor. 3.—If any line be drawn parallel to PP' or EE', it is obvious that the portion intercepted between the asymptotes will be bisected by the diameter CD; but the segments intercepted between the curve and the asymptotes are also equal, and hence *any diameter bisects all its ordinates.*

Cor. 4. $$PQ \times PQ' = DE^2.$$

For, if through D a perpendicular be drawn to the axis, meeting the asymptotes in K, K'.

Then $DK : DE = PR : PQ;$ (Similar $\triangle^s$.)

also $DK' : DE' = Pr : PQ';$ (Do.   )

$\therefore\ DK \times DK' : DE \times DE' = PR \times Pr : PQ \times PQ';$

but $DK \times DK' = BC^2 = PR \times Pr;$ (Prop. XVII.)

$$\therefore\ PQ \times PQ' = DE \times DE'$$

$$= DE^2.   \text{(See *Cor.* 1.)}$$

Cor. 5.—If through any point on a hyperbola a line be drawn in a constant direction, the rectangle under the segments intercepted between the point and the asymptotes is constant.

PROPOSITION XIX.

If CP' be the semi-diameter conjugate to CP, and tangents be drawn at P, P', meeting the transverse and conjugate axes in T, T' respectively; TQ, $T'Q'$ perpendiculars to the axes meeting the circles described on the axes as diameters in Q, Q'; then $\angle QCA = \angle Q'CB$.
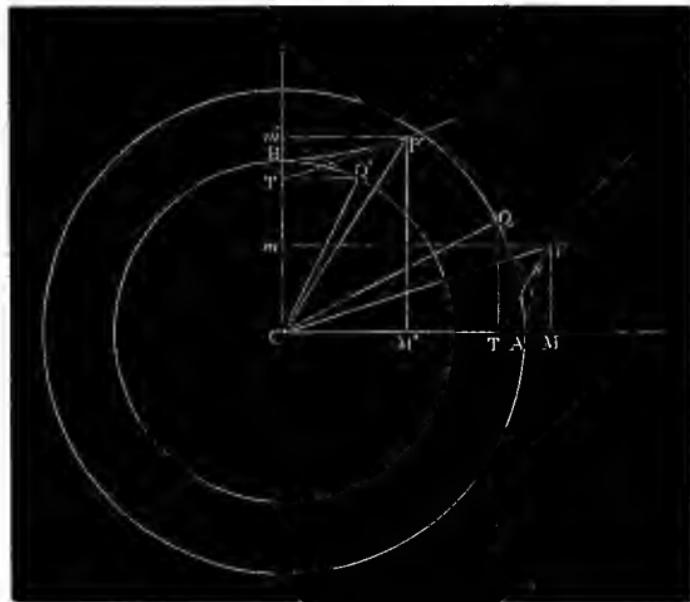

Fig. 24.

Join QM, $Q'm'$.

Then QM, $Q'm'$ are tangents to the circles at Q, Q', respectively.   (*Cor.* 1, Prop. IX.)

$$PM : MQ = CB : CA, \quad (Cor. \ 2, \ \text{Prop. IX.})$$

and $\qquad P'm' : m'Q' = CA : CB;$

$$\therefore P'm' \times PM = Q'm' \times QM.$$

But since CP' is $\parallel PT$;

$$\therefore PM : MT = Cm' : m'P' ;$$
$$\therefore PM \times m'P' = Cm' \times MT ;$$
$$\therefore Q'm' \times QM = Cm' \times MT ;$$
$$\therefore Cm' : m'Q' = QM : MT$$
$$= CM : MQ ;$$

<div align="right">(8, VI. Euclid.)</div>

hence the $\triangle^s$ $m'CQ'$ and MCQ are similar ;

<div align="right">(7, VI. Euclid.)</div>

and $\qquad \therefore \angle QCA = \angle Q'CB.$

Cor. 1.—CP is parallel to the tangent at P'.

$$\text{For} \quad P'm' : m'T' = \begin{Bmatrix} P'm' : m'Q' \\ m'Q' : m'T' \end{Bmatrix}$$
$$= \begin{Bmatrix} CA : CB \\ Cm' : m'Q' \end{Bmatrix} \quad \begin{matrix} (Cor.\ 2,\ \text{Prop. ix.}) \\ (8,\ \text{VI. Euclid.}) \end{matrix}$$
$$= \begin{Bmatrix} QM : MP \\ CM : QM \end{Bmatrix} \quad \begin{matrix} (Cor.\ 2,\ \text{Prop. ix.}) \\ (\text{Prop. xix.}) \end{matrix}$$
$$= CM : MP = Pm : Cm ;$$

$\therefore \triangle^s$ $P'm'T'$ and CPm are similar, and $\therefore CP \parallel P'T'$.

Hence, if one diameter CP' of an hyperbola be conjugate to another CP, then, conversely, CP will be conjugate to CP'.

Cor. 2. $\quad PM = Q'm' = Cm$; and $P'm' = QM = CM'$.

Since the $\triangle^s$ $Cm'Q'$ and CMQ are similar,

$$m'Q' : MQ = CQ' : CQ$$
$$= CB : CA.$$

But $\qquad PM : MQ = CB : CA ;$ (*Cor.* 2, Prop. ix.)

$$\therefore m'Q' : MQ = PM : MQ ;$$
$$\therefore m'Q' = PM.$$

Also $\qquad P'm' : m'Q' = CA : CB$
$$= QM : Q'm' ; \quad (\text{Similar } \triangle^s.)$$
$$\therefore P'm' = QM.$$

Cor. 3. $CM^2 - CM'^2 = CA^2.$

For $CA^2 = CQ^2 = CM^2 - QM^2 = CM^2 - CM'^2.$ (*Cor.* 2.)

Cor. 4. $P'M'^2 - PM^2 = CB^2.$

For $CB^2 = CQ'^2 = Cm'^2 - m'Q'^2 = P'M'^2 - PM^2.$ (*Cor.* 2.)

Cor. 5.—The difference of the squares of any pair of conjugate semi-diameters is equal to the difference of the squares of the semiaxes.

For
$$CA^2 - CB^2 = CM^2 - CM'^2 - (P'M'^2 - PM^2)$$
$$\text{(\textit{Cors.} 3 and 4.)}$$
$$= CM^2 + PM^2 - (CM'^2 - P'M'^2)$$
$$= CP^2 - CP'^2.$$

Cor. 6. $PM : CM' = CB : CA = P'M' : CM.$

For $PM : QM = CB : CA;$ (*Cor.* 2, Prop. IX.)

but $QM = m'P' = CM';$

$\therefore PM : CM' = CB : CA.$

Again, $Cm' : CM = CQ' : CQ;$ (Similar $\triangle^s$.)

$\therefore P'M' : CM = CB : CA.$

Cor. 7.—The $\triangle^s$ CPM and $CP'M'$ are equal in area.

For $PM \times CM = P'M' \times CM'.$ (*Cor.* 6.)

Cor. 8. Mm' is parallel to an asymptote.

For $Cm' : Cm = CQ' : CQ$ (Similar $\triangle^s$.)
$$= CB : CA;$$

$\therefore Mm'$ is $\parallel AB$, and $\therefore \parallel$ an asymptote.
(*Cor.* 7, Prop. IV.)

Cor. 9.—If MP, $m'P'$ be produced to meet in O, then CO will be an asymptote.

Since $CMOm'$ is a parallelogram, CO will bisect Mm', and $\therefore$ it will bisect AB, which is $\parallel Mm'$.

Hence CO is an asymptote.   (*Cor.* 7, Prop. IV.)

PROPOSITION XX.

If CP, CP' be conjugate semi-diameters, PM, PM' ordinates to the axis, PT a tangent at P; then

$$\triangle\ CVM' = \triangle\ PTM, \text{ and } \triangle\ CP'V = \triangle\ CPT.$$

Fig. 25.

For $\triangle\ CPM : \triangle\ CHA : \triangle\ CVM' = CM^2 : CA^2 : CM'^2;$
$$(19, \text{VI. Euclid.})$$

$\therefore\ \triangle\ CPM - \triangle\ CHA : \triangle\ CVM' = CM^2 - CA^2 : CM'^2;$

but $\qquad\qquad CM^2 - CA^2 = CM'^2;$ (*Cor.* 3, Prop. XIX.)

$\qquad\therefore\ \triangle\ CPM - \triangle\ CHA = \triangle\ CVM',$

or $\qquad\qquad$ trap. $PMAH = \triangle\ CVM';$

$$\therefore\ \triangle\ PTM = \triangle\ CVM'.$$
$$(Cor., \text{Prop. XVI.})$$

Again, $\qquad\qquad \triangle\ CPM = \triangle\ CP'M';$
$$(Cor.\ 7, \text{Prop. XIX.})$$

$\qquad\therefore\ \triangle\ CPM - \triangle\ PTM = \triangle\ CP'M' - \triangle\ CVM';$

$$\therefore\ \triangle\ CP'V = \triangle\ CPT.$$

Proposition XXI.

If QG be an ordinate drawn to any diameter Pp, from any point Q on a hyperbola, cutting the transverse axis in L ; QS an ordinate drawn to the axis from the same point and produced, if necessary, to meet the diameter Pp in K ; AH a tangent at the vertex ; then $\triangle\, QLS =$ trap. $KSAH$.
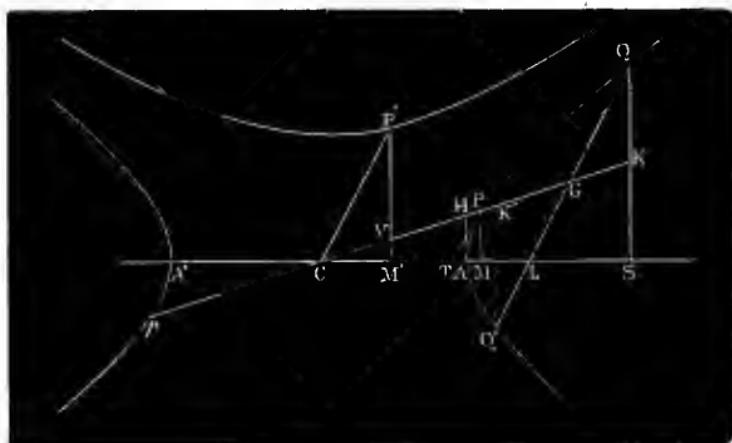

Fig. 26.

Let CP' be the diameter conjugate to CP; draw the ordinates PM, $P'M'$ to the axis.

Then $CA : CS : CM = AH : SK : MP$;

$$\text{(Similar } \Delta^s.)$$

$\therefore\ CA + CS : CA + CM = AH + SK : AH + MP$,

or $A'S : A'M = AH + SK : AH + MP$;

$\therefore\ A'S \times SA : A'M \times MA = (AH + SK)\, SA : (AH + MP)\, MA$;

$\therefore\ QS^2 : PM^2 = $ trap. $KSAH$: trap. $PMAH$;

$$\text{(Prop. x.)}$$

$$\therefore \Delta QSL : \Delta PMT = \text{trap. } KSAH : \text{trap. } PMAH,$$

(19, VI. Euclid.)

but $\Delta PMT = \text{trap. } PMAH;$ (*Cor.* Prop. xvi.)

$$\therefore \Delta QSL = \text{trap. } KSAH.$$

Cor. 1. $\Delta QGK = \text{trap. } TPGL.$

For $\Delta QSL = \text{trap. } KSAH.$

$$\therefore \Delta QGK = \text{trap. } GLAH = \text{trap. } TPGL.$$

(Prop. xvi.)

Cor. 2.—In like manner it can be shown that

$$\Delta GQ'K' = \text{trap. } TPGL;$$

$$\therefore \Delta GQK = \Delta GQ'K';$$

but they are also similar; $\therefore GQ = GQ'.$

Hence, any diameter bisects all chords parallel to the tangents at its extremities. (See also *Cor.* 3, Prop. xviii.)

PROPOSITION XXII.

If CP' be the semi-diameter conjugate to CP, QG any ordinate to CP; then $QG^2 : PG \times Gp = CP'^2 : CP^2$. (See Fig. 26.)

For $\Delta CPT : \Delta CGL = CP^2 : CG^2.$

(19, VI. Euclid.)

Division, $\Delta CPT : \text{trap. } TPGL = CP^2 : CG^2 - CP^2,$

or $\Delta CP'V : \Delta QGK = CP^2 : PG \times Gp;$

(Prop. xx.)

$$\therefore CP'^2 : QG^2 = CP^2 : PG \times Gp;$$

(19, VI. Euclid.)

hence $QG^2 : PG \times Gp = CP'^2 : CP^2.$

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Cor. 1.—Ordinates to any diameter at equal distances from the centre are equal.

For the rectangles are the same for equal distances from the centre.

Cor. 2.— $\quad QG^2 : CG^2 - CP^2 = CP'^2 : CP^2.$

For $\qquad PG \times Gp = CG^2 - CP^2.$

Cor. 3.—If the ordinate GQ be produced to meet the conjugate hyperbola in Q'; then

$$Q'G^2 : CG^2 + CP^2 = CP'^2 : CP^2.$$

Fig. 27.

Draw $Q'G'$ an ordinate to CP'.

Then $\quad Q'G'^2 : CG'^2 - CP'^2 = CP^2 : CP'^2.$ $\qquad$ (*Cor.* 2.)

Alt. $\qquad CG^2 : CP^2 = CG'^2 - CP'^2 : CP'^2;$

$\therefore\ CG^2 + CP^2 : CP^2 = CG'^2 : CP'^2;$

$\therefore\ Q'G^2 : CG^2 + CP^2 = CP'^2 : CP^2$

Cor. 4.—The squares of the ordinates to any diameter are proportional to the rectangles under the segments of that diameter.

Proposition XXIII.

If a normal at any point P of a hyperbola meet the transverse axis in N, and the conjugate axis in N'; then

$$PN : CP' = CB : CA = CP' : PN',$$

CP' being the semi-diameter conjugate to CP.
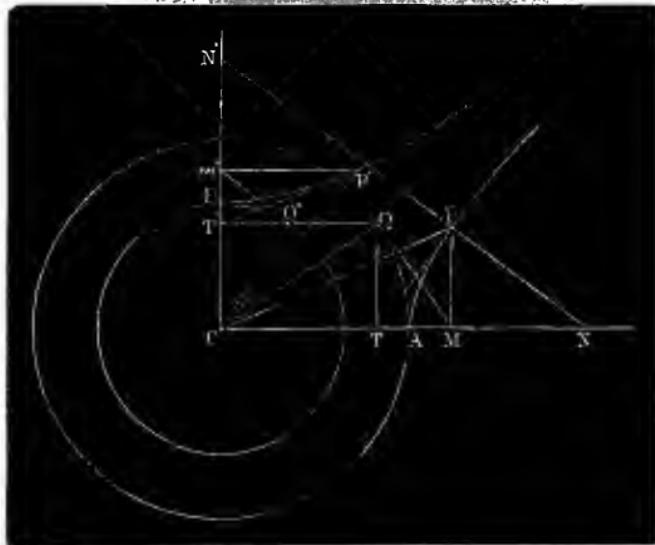
Fig. 28.

Since CP' is $\parallel PT$, the $\triangle P'Cm'$ is similar to $\triangle PMT$, and $\therefore$ similar to $\triangle NPM$;

$$\therefore P'C : NP = P'm' : PM$$
$$= MQ : PM \quad (Cor.\ 2,\ \text{Prop. xix.})$$
$$= CA : CB. \quad (Cor.\ 2,\ \text{Prop. ix.})$$

Also $\quad PN' : PN = CM : MN \quad$ (Parallel lines.)
$$= CA^2 : CB^2; \quad (Cor.,\ \text{Prop. viii.})$$

$\therefore PN' \times PN : PN^2 = CA^2 : CB^2 = P'C^2 : PN^2$;

hence $\quad PN \times PN' = P'C^2,$

or $\quad PN' : P'C = P'C : PN = CA : CB.$

Cor.— $\quad PN \times PN' = P'C^2.$

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PROPOSITION XXIV.

The rectangle under the distances of the foci, from any point P on a hyperbola, is equal to the square of the semi-diameter CP' conjugate to that passing through the point.

Fig. 29.

Circumscribe the $\triangle FPF'$ by a circle, cutting the conjugate axis in L and N'; join PN', FN'; produce PN' to meet the transverse axis in N.

Then, since LN' bisects FF', and is also perpendicular to it; $\therefore LN'$ is the diameter of the circle, and arc FN' = arc $F'N'$;

hence $\angle F'PL = FPL$; $\therefore PL$ is a tangent.　(Prop. III.)

Also $\angle LPN' = 90°$; $\therefore PN$ is a normal.

Now $\angle FPN = \angle F'PN'$, and $\angle PFN = \angle PN'F$;
(26, III. Euclid.)

$\therefore$ the $\triangle^s FPN$ and $F'PN'$ are similar;

$\therefore F'P : PN' = PN : PF$;

$\therefore PF \times PF' = PN \times PN' = CP'^2.$
(*Cor.*, Prop. XXIII.)

Otherwise thus—

$$2CA = PF' - PF;$$
$$\therefore\ 4CA^2 = PF'^2 + PF^2 - 2PF' \times PF$$
$$\text{(4, II. Euclid.)}$$
$$= 2CP^2 + 2CF^2 - 2PF' \times PF;$$
$$\text{(Cor., 13, II. Euclid.)}$$
$$\therefore\ PF' \times PF = CF^2 + CP^2 - 2CA^2$$
$$= CF^2 - CA^2 + CP^2 - CA^2$$
$$= CB^2 + CP^2 - CA^2\ \text{(Cor. 5, Prop. iv.)}$$
$$= CP^2 - CP^2 + CP'^2\ \text{(Cor. 5, Prop. xix.)}$$
$$= CP'^2.$$

Cor. 1.—Since PL bisects the $\angle FPF'$, it is a tangent, and $\therefore$ PN, which is $\perp PL$, is a normal. Hence the circle which passes through the foci and any point P on a hyperbola, passes also through the points in which the tangent and normal at P intersect the conjugate axis.

Cor. 2.—If K be the foot of the perpendicular let fall from N' on either PF' or PF produced; then

$$PK = \tfrac{1}{2}(PF' - PF) = \text{semi-transverse axis.}$$

Cor. 3.—If K' be the foot of the perpendicular let fall from L on PF'; then

$$F'K' = \tfrac{1}{2}(PF' - PF) = \text{semi-transverse axis.}$$

Cor. 4.— $\quad CA : CB = CP' : PN.$

(See Fig. 11.)

For $\quad F'P : PF = F'N : NF.$ (Prop. B., VI. Euclid.)

Div. $\quad F'P - PF : PF = F'N - NF : NF.$

Alt. $\qquad 2CA : 2CF = PF : NF.$

Similarly, $2CA : 2CF = PF' : NF'$;

$$\therefore\ CA^2 : CF^2 = PF \times PF' : NF \times NF'.$$

$$\therefore\ CA^2 : CF^2 - CA^2 = PF \times PF' : NF \times NF' - PF \times PF';$$

$$\therefore\ CA^2 : CB^2 = CP'^2 : PN^2.$$

PROPOSITION XXV.

If CK be a perpendicular let fall from the centre on the tangent to a hyperbola at any point P, and CP' the semi-diameter conjugate to CP; then $CK : CA = CB : CP'$.
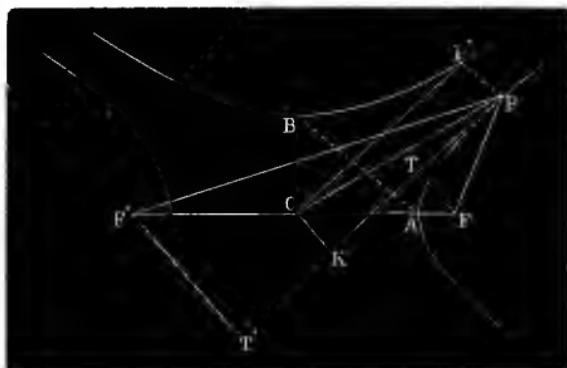

Fig. 30.

Draw FT, $F'T' \perp$ the tangent at P.

Since the $\triangle^s FPT$ and $F'PT'$ are similar;

$$\therefore FP : F'P = FT : F'T' = F'P - FP : F'T' - FT$$
$$= CA : CK;$$

$$\therefore FP \times F'P : FT \times F'T' = CA^2 : CK^2,$$

but $\qquad\qquad FP \times F'P = CP'^2,$ (Prop. xxiv.)

and $\qquad\qquad FT \times F'T' = CB^2;$ (Prop. vi.)

$$\therefore CP'^2 : CB^2 = CA^2 : CK^2;$$

$$\therefore CP' : BC = CA : CK.$$

Cor.—The area of the triangle formed by joining the extremities of any pair of conjugate semi-diameters is constant.

For $\qquad\qquad CP' \times CK = CB \times CA;$

$\therefore$ area of $\triangle P'CP$ = area of $\triangle BCA$, and therefore constant.

PROPOSITION XXVI.

If a chord PP' of a hyperbola pass through a fixed point O, the rectangle under its segments is to the square of the parallel semi-diameter CR in a constant ratio.

Fig. 31.

Draw the diameter passing through O, meeting the curve in S; draw the ordinates SM, SM'.

Then $PN^2 : CR^2 = CN^2 - CR'^2 : CR'^2$.   (Prop. XXII.)

Comp. $PN^2 + CR^2 : CR^2 = CN^2 : CR'^2$.

Alt. $PN^2 + CR^2 : CN^2 = CR^2 : CR'^2$.

Similarly, $SM^2 + CR^2 : CM^2 = CR^2 : CR'^2$;

$\therefore PN^2 + CR^2 : SM^2 + CR^2 = CN^2 : CM^2$

$$= ON^2 : SM^2 ;$$

(Similar $\triangle^s$.)

$\therefore PN^2 + CR^2 - ON^2 : ON^2 = SM^2 + CR^2 - SM^2 : SM^2 ;$

$\therefore CR^2 + PO \times OP' : ON^2 = CR^2 : SM^2$.

Alt. $CR^2 + PO \times OP' : CR^2 = ON^2 : SM^2$

$$= CO^2 : CS^2.$$

Division, $PO \times OP' : CR^2 = CO^2 - CS^2 : CS^2$.

CASE 2.—(The importance of this proposition renders it advisable to give a separate demonstration of each case).

As before, draw the diameter passing through O; also PN' and SM ordinates to the diameters CR and CR' respectively.
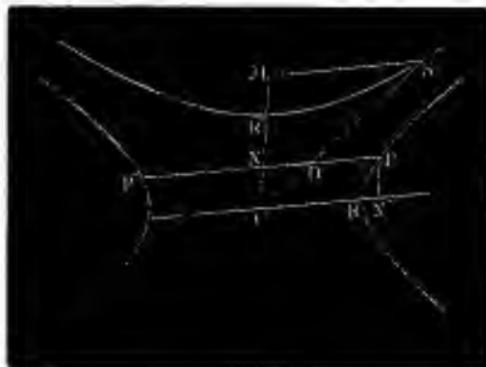
Fig. 32.

$$SM^2 : CM^2 - CR'^2 = CR^2 : CR'^2 ; \quad \text{(Prop. xxii.)}$$

$$\therefore \ SM^2 + CR^2 : CM^2 = CR^2 : CR'^2.$$

But $\quad CN'^2 - CR^2 : PN'^2 = CR^2 : CR'^2 ; \quad \text{(Prop. xxii.)}$

$$\therefore \ SM^2 + CR^2 : CN'^2 - CR^2 = CM^2 : PN'^2,$$

or $\qquad SM^2 + CR^2 : PN^2 - CR^2 = CM^2 : CN^2$

$$= SM^2 : ON^2 ;$$
$$\text{(Similar } \triangle^s.)$$

$\therefore \ SM^2 + CR^2 - SM^2 : SM^2 = PN^2 - ON^2 - CR^2 : ON^2,$

or $\qquad CR^2 : PO \times OP' - CR^2 = SM^2 : ON^2$

$$= CS^2 : CO^2 ;$$

$$\therefore \ PO \times OP' : CR^2 = CS^2 + CO^2 : CS^2.$$

CASE 3.—Same construction.

Fig. 33.

$$PN'^2 : CN'^2 - CR^2 = CR'^2 : CR^2. \quad \text{(Prop. XXII.)}$$

Also $\quad SM'^2 : CM'^2 - CR^2 = CR'^2 : CR^2; \quad \text{(Prop. XXII.)}$

$$\therefore \; CM'^2 - CR^2 : CN'^2 - CR^2 = SM'^2 : PN'^2,$$

or $\qquad SM^2 - CR^2 : PN^2 - CR^2 = CM^2 : CN^2$

$$= SM^2 : ON^2;$$

$$\therefore \; SM^2 + CR^2 - SM^2 : ON^2 - PN^2 + CR^2 = SM^2 : ON^2,$$

or $\qquad CR^2 : OP \times OP' + CR^2 = CS^2 : CO^2;$

$$\text{(Similar } \triangle^s.)$$

$$\therefore \; OP \times OP' : CR^2 = CO^2 - CS^2 : CS^2.$$

Cor. 1.—If two chords PP', pp' of a hyperbola intersect, the rectangles under the segments are proportional to the squares of the parallel semi-diameters.

Let Cr be the semi-diameter $\parallel pp'$.

Then $OP \times OP' : CR^2 = CO^2 - CS^2 : CS^2 = Op \times Op' : Cr^2;$

$$\therefore \; OP \times OP' : Op \times Op' = CR^2 : Cr^2.$$

Cor. 2.—If from any point two tangents be drawn to a hyperbola, the tangents will be proportional to the parallel semi-diameters.

Proposition XXVII.

If from a point without a hyperbola two tangents OP, OR be drawn, any line AQ' drawn parallel to either will be cut by the curve and chord of contact of the tangents in geometric proportion.
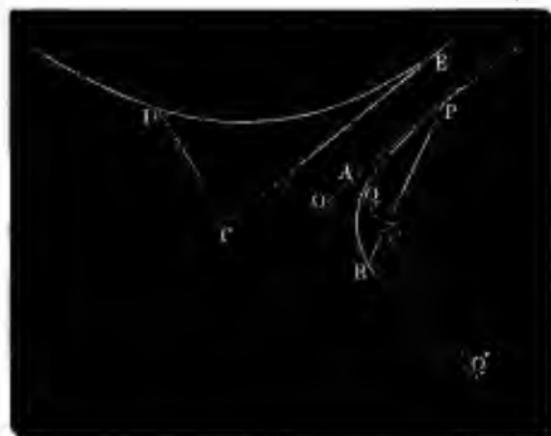

Fig. 34.

Draw the semi-diameters CE and $CD \parallel OP$ and OR respectively.

Then $AQ \times AQ' : AP^2 = CD^2 : CE^2$

$(Cor.$ 1, Prop. xxvi.$)$

$= OR^2 : OP^2$

$(Cor.$ 2, Prop. xxvi.$)$

$= AB^2 : AP^2$;   (Similar Δ^s.)

$\therefore AQ \times AQ' = AB^2$.

PROPOSITION XXVIII.

If a circle intersect a hyperbola in four points, the common chords will be equally inclined to the axis.
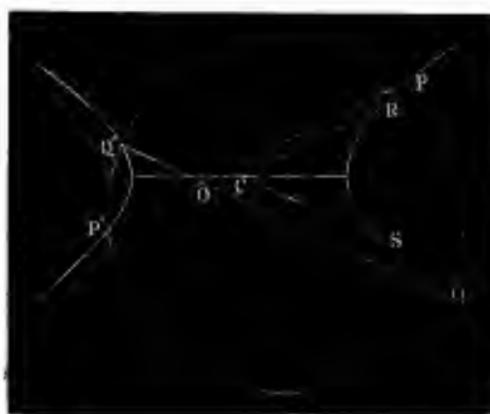

Fig. 35.

Let PP', QQ' be the common chords intersecting in O.

Draw the semi-diameters CR, $CS \parallel PP'$, QQ' respectively.

Then $OP \times OP' : OQ \times OQ' = CR^2 : CS^2$;

(*Cor.* 1, Prop. XXVI.)

but $OP \times OP' = OQ \times OQ'$; (35, III. Euclid.)

$\therefore CR = CS.$

Hence CR and CS, and $\therefore PP'$ and QQ', are equally inclined to the axis. (*Cor.* 1, Prop. I.)

Cor.—In like manner it can be shown that the common chords PQ and $P'Q'$, also PQ' and $P'Q$, are equally inclined to the axis.

PROPOSITION XXIX.

If a tangent to a hyperbola at any point P meet *any* diameter AA' in T, and the ordinate PM be drawn; then CA is a mean proportional between CM and CF.
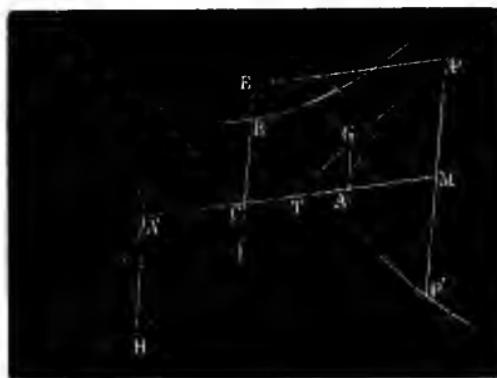

Fig. 36.

Draw tangents at A and A', meeting the tangent at P in G and H.

Then $\qquad A'T : TA = A'H : AG,$ (Similar $\triangle^s$.)

and $\qquad A'M : MA = HP : PG$; (Parallel lines.)

but $\qquad A'H : AG = HP : PG$;

$\qquad\qquad\qquad\qquad$ (*Cor.* 2, Prop. XXVI.)

$$\therefore A'T : TA = A'M : MA;$$

$$\therefore A'T + TA : A'T - TA = A'M + MA : A'M - MA,$$

or $\qquad\qquad 2CA : 2CT = 2CM : 2CA.$

hence $\qquad\qquad CM \times CT = CA^2.$

Cor. 1.—Conversely, if PM be an ordinate to any diameter AA', and CT be taken a third proportional to CM and CA; then PT will be a tangent to the hyperbola.

Cor. 2.—The tangents at the extremities of any double ordinate intersect on the diameter corresponding to that ordinate.

Cor. 3.—If a diameter be drawn through the intersection of two tangents to a hyperbola, it will bisect the chord of contact.

Cor. 4.—If the tangent at P meet a diameter of the conjugate hyperbola in D, and the ordinate PE be drawn; then CB is a mean proportional between CD and CE.

For $\qquad CT : CM = TD : DP \qquad$ (Parallel lines.)

$$= DC : DE; \qquad \text{(Similar } \Delta^s.)$$

$\qquad \therefore CT \times CM : CM^2 = DC : DE$

or $\qquad\qquad CA^2 : CM^2 = DC : DE;$

$\qquad \therefore CA^2 : CM^2 - CA^2 = DC : DE - DC;$

$\qquad \therefore CB^2 : PM^2 \text{ or } CE^2 = DC : CE$

$$\text{(}Cor.\text{ 2, Prop. xxii.)}$$

$$= DC \times CE : CE^2;$$

hence $\qquad\qquad CB^2 = DC \times CE.$

Cor. 5.—Any diameter is cut harmonically by a tangent and the ordinate to the diameter drawn from the point of contact of the tangent.

Since $\qquad CM : CA = CA : CT.$

Comp. and Div.

$$CM + CA : CM - CA = CA + CT : CA - CT,$$

or $\qquad\qquad A'M : MA = A'T : TA.$

Cor. 6.—Any tangent to a hyperbola will be cut harmonically by two parallel tangents and the diameter passing through their points of contact.

For $\qquad MA' : MT : MA = PH : PT : PG.$

$$\text{(2, VI. Euclid.)}$$

PROPOSITION XXX.

If a variable tangent to a hyperbola meet two fixed parallel tangents, it will intercept segments on them whose rectangle is constant, and equal to the square of the parallel semi-diameter.

See Fig. 36.

Let PGH be the variable tangent meeting the fixed tangents AG, $A'H$ in the points G, H.

Draw the semi-diameter $CB \parallel AG$ or $A'H$; also draw PE an ordinate to BC.

Then $CT : CA = CA : CM.$ (Prop. XXIX.)

Division, $CT : TA = CA : AM.$

Alt. $CT : CA = TA : AM.$

Comp. $CT : TA' = TA : TM;$

$\therefore CD : A'H = AG : PM;$ (Similar $\triangle^s$.)

$\therefore A'H \times AG = CD \times PM$

$= CD \times CE$

$= CB^2.$ (*Cor.* 4, Prop. XXIX.)

Cor.—The rectangle under the segments of the variable tangent is equal to the square of the semi-diameter CQ drawn parallel to it.

For $A'H : HP = CB : CQ = AG : PG;$
$(Cor.$ 2, Prop. XXVI.)

$\therefore A'H \times AG : HP \times PG = CB^2 : CQ^2.$

But $A'H \times AG = CB^2;\ \therefore HP \times PG = CQ^2.$

Proposition XXXI.

The triangles CPT and CAK, formed by drawing tangents at the extremities of any two semi-diameters of a hyperbola, are equal in area.
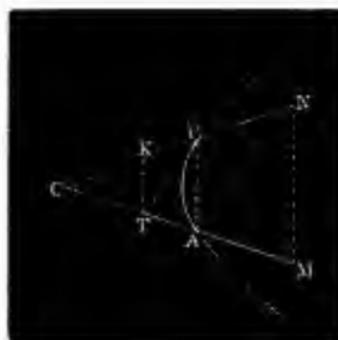

Fig. 37.

Draw the ordinate PM.

Then $\qquad TC : CA = CA : CM$; $\qquad$ (Prop. xxix.)

$\qquad\qquad\qquad\qquad = CK : CP$; $\qquad$ (Similar $\triangle$'s.)

$\qquad\quad \therefore KT$ is $\parallel PA$; $\qquad$ (2, VI. Euclid.)

$\qquad\quad \therefore \triangle PTA = \triangle PKA$; $\qquad$ (37, I. Euclid.)

hence $\qquad\qquad \triangle CPT = \triangle CAK$.

Cor.—If the ordinate AN be drawn from A to the semi-diameter CP produced; then

$\qquad$ Area of $\triangle CMP$ = area of $\triangle CAN$.

For $\quad AN$ is $\parallel PT$; $\therefore TC : CA = PC : CN$;

hence $\qquad CA : CM = PC : CN$;

$\qquad\quad \therefore MN$ is $\parallel PA$; $\qquad$ (2, VI. Euclid.)

$\qquad\quad \therefore \triangle PAM = \triangle PAN$; $\qquad$ (31, I. Euclid.)

hence $\qquad\qquad \triangle CMP = \triangle CAN$.

PROPOSITION XXXII.

If from the extremities of any two conjugate diameters CP, CQ, the ordinates PM, QN be drawn to *any* other diameter CA ; then

$$CN^2 = AM \times MA' ; \text{ and } CM^2 = AM \times MA' + CA^2.$$
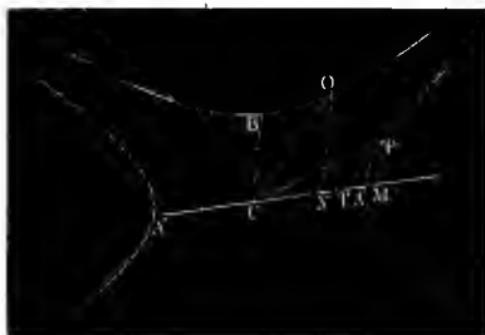
Fig. 38.

For $\quad PM^2 : CM^2 - CA^2 = CB^2 : CA^2$

(*Cor.* 2, Prop. XXII.)

and $\quad QN^2 : CN + CA^2 = CB^2 : CA^2$;

(*Cor.* 3, Prop. XXII.)

$\therefore PM^2 : QN^2 = CM^2 - CA^2 : CN^2 + CA^2$;

but $\quad PM : QN = TM : CN$; (Similar $\triangle^s$.)

$\therefore TM^2 : CN^2 = CM^2 - CA^2 : CN^2 + CA^2$

$= CM^2 - CT \times CM : CN^2 + CA^2$

(Prop. XXIX.)

$= CM \times TM : CN^2 + CA^2$;

(2, II. Euclid.)

$\therefore TM^2 : CM \times TM = CN^2 : CN^2 + CA^2$;

$\therefore TM : CM = CN^2 : CN^2 + CA^2$;

Div. $\quad TM : CT = CN^2 : CA^2$;

$$\therefore\ TM \times CM : CT \times CM = CN^2 : CA^2\ ;$$

but $$CT \times CM = CA^2\ ;\qquad \text{(Prop. xxix.)}$$

$$\therefore\ TM \times CM = CN^2\ ;$$

hence $$CN^2 = CM^2 - CM \times CT\quad \text{(2, II. Euclid.)}$$

$$= CM^2 - CA^2$$

$$= AM \times MA'.$$

Also $$CM^2 = CN^2 + CA^2$$

$$= AM \times MA' + CA^2.$$

Cor. 1.— $$CM^2 - CN^2 = CA^2.$$

Cor. 2.— $$QN^2 - PM^2 = CB^2.$$

For $$QN^2 : (CN^2 + CA^2)\ \text{or}\ CM^2 = CB^2 : CA^2,$$

and $$PM^2 : CM^2 - CA^2 = CB^2 : CA^2\ ;$$
$$\text{(Cor. 2, Prop. xxii.)}$$

$$\therefore\ QN^2 - PM^2 : CA^2 = CB^2 : CA^2\ ;$$

hence $$QN^2 - PM^2 = CB^2.$$

Cor. 3.— $$CN : PM = CA : CB = CM : QN.$$

For $$CA^2 : CB^2 = CM^2 - CA^2 : PM^2 = CN^2 + CA^2 : QN^2,$$

or $$CA^2 : CB^2 = CN^2 : PM^2 = CM^2 : QN^2\ ;\quad \text{(Cor. 1.)}$$

$$\therefore\ CA : CB = CN : PM = CM : QN.$$

Cor. 4.—The $\triangle\ CMP = \triangle\ CNQ$ in area.

For $$CN : PM = CM : QN,\qquad \text{(Cor. 3.)}$$

and $$\angle\ CMP = \angle\ CNQ\ ;$$

$$\therefore\ \triangle\ CMP = \triangle\ CNQ.\quad \text{(15, VI. Euclid.)}$$

M

PROPOSITION XXXIII.

If any tangent to a hyperbola meet any two conjugate diameters CP, CQ, the rectangle under its segments is equal to the square of the parallel semi-diameter CB.
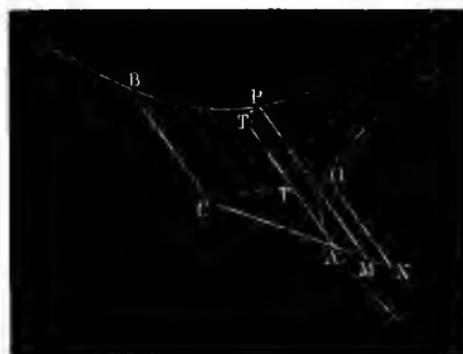

Fig. 39.

Draw the ordinates PM, QN, to the diameter passing through the point of contact.

Then $\quad\quad CM : PM = CA : AT'$;	(Similar $\triangle^s$.)

$\quad\quad \therefore\ CM \times AT' = PM \times CA$

$\quad\quad\quad\quad\quad\quad\quad = CB \times CN$;

$\quad\quad\quad\quad\quad\quad\quad$ (*Cor.* 3, Prop. xxxii.)

$\quad\quad \therefore\ CM : CN = CB : AT'$.

Again, $\quad\quad CN : NQ = CA : AT$.	(Similar $\triangle^s$.)

$\quad\quad \therefore\ CN \times AT = CA \times NQ$

$\quad\quad\quad\quad\quad\quad\quad = CB \times CM$;

$\quad\quad\quad\quad\quad\quad\quad$ (*Cor.* 3, Prop. xxxii.)

$\quad\quad \therefore\ CM : CN = AT : CB$;

hence $\quad\quad CB : AT' = AT : CB$;

$\quad\quad \therefore\ AT \times AT' = CB^2$.

PROPOSITION XXXIV.

Given in magnitude and position any two conjugate semi-diameters CA', CB', of a hyperbola, to find the axes.
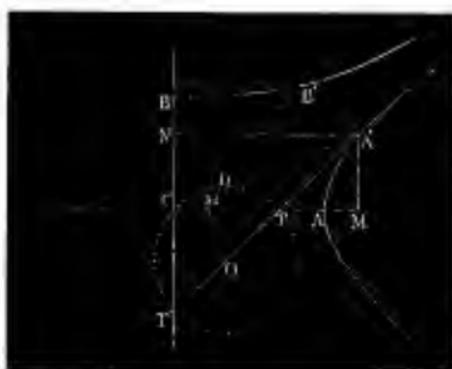

Fig. 40.

Take $A'D$ a third proportional to CA' and CB'.

Bisect CD in H. Draw $HO \perp CD$, meeting $A'O$ drawn $\parallel CB'$ in O. With the centre O, and radius OC, describe a circle cutting OA' in T, T'; join CT, CT'; draw $A'M$, $A'N \perp CT$, CT', produced respectively. Take CA a mean proportional between CT and CM; also CB a mean proportional between CT' and CN. Then CA, CB are the axes.

For $$CB'^2 = CA' \times A'D \qquad \text{(Const.)}$$

$$= TA' \times A'T' ; \quad (36, \text{III. Euclid.})$$

$\therefore$ CA and CB are conjugate diameters; (Prop. XXXIII.)

but $\angle TCT' = 90°$; $\therefore$ CA and CB are the axes in position.

Also $CA^2 = CT \times CM$, and $CB^2 = CT' \times CN$;

$\therefore$ CA and CB represent the semiaxes in magnitude.

(Prop. IX.)

M 2

PROPOSITION XXXV.

If any line *TS* be drawn parallel to the chord of contact of two tangents to a hyperbola, the segments *AT, BS*, intercepted between the curve and the tangents will be equal.
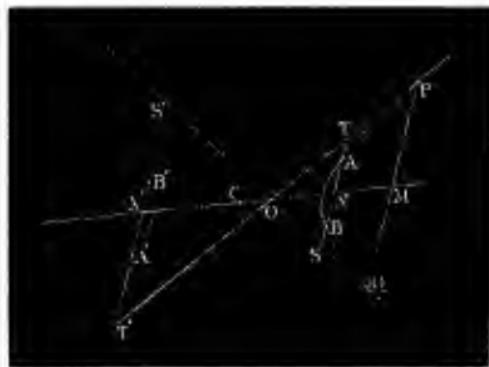

Fig. 41.

Draw the diameter of the parallel chords *BA, PQ*.
This diameter will pass through *O*.

<div align="right">(Cor. 2, Prop. XXIX.)</div>

Then, since $\qquad PM = MQ;$ (*Cor.* 2, Prop. XXI.)

$$\therefore TN = NS;$$

but $\qquad\qquad AN = NB;$ (*Cor.* 2, Prop. XXI.)

hence $\qquad\qquad AT = BS.$

Cor. 1.—The tangent drawn parallel to the chord of contact of two other tangents is bisected at its point of contact.

Cor. 2.—If a line be drawn parallel to the chord of contact of two tangents, the segment intercepted between the tangents is bisected by the diameter passing through their intersection.

PROPOSITION XXXVI.

Any line *OA* drawn through the intersection of two tangents to a hyperbola is cut harmonically by the curve and the chord of contact of the tangents.
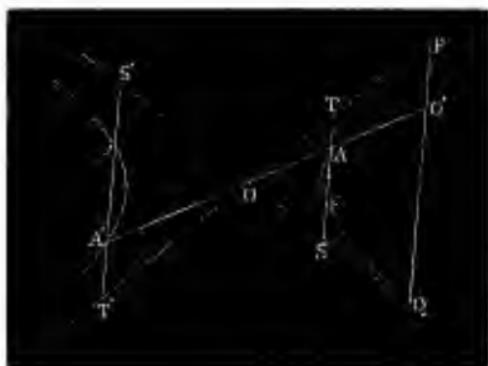

Fig. 42.

Through *A* and *A'* draw *TS, T'S'* ‖ *PQ*, the chord of contact of the tangents. Then

$$TP^2 : T'P^2 = TA \times TB : T'A' \times T'B';$$
$$(\textit{Cor.}, \text{ Prop. xxvi.})$$

but $TB = AS$, and $T'B' = A'S'$;   (Prop. xxxv.)

$$\therefore\ TP^2 : T'P^2 = TA \times AS : T'A' \times A'S';$$

but $TA : T'A' = OA : OA' = AS : AS'$;
$$(\text{Similar } \triangle^s.)$$

$$\therefore\ TP^2 : T'P'^2 = OA^2 : OA'^2;$$

$$\therefore\ TP : T'P = OA : OA';$$

hence $AO' : A'O = OA : OA'.$ (Parallel lines.)

PROPOSITION XXXVII.

Any line drawn through the middle point of the chord of contact of two tangents to a hyperbola will be cut harmonically by the curve and the line drawn through the intersection of the tangents parallel to their chord of contact.

Fig. 43.

Let TP, TQ, be tangents, BB' any chord drawn through O', the middle point of PQ, and meeting the line drawn through $T \parallel PQ$ in O. Through B'' draw $B''B' \parallel$ to PQ; join TC.

Then TC produced will pass through O'.

(Cor. 3, Prop. XXIX.)

$$B''M : B'B'' = \begin{cases} B''M : O'T' \\ O'T' : B'B'' \end{cases}$$

$$= \begin{cases} B''T : TT' \\ BT' : BB'' \end{cases} \qquad \text{(Similar } \triangle^s.)$$

$$= B''T \times BT' : BB'' \times TT'.$$

But $\qquad TT' \times BB'' = 2B''T \times BT;$

$$\therefore B'B'' = 2B''M;$$

$\therefore B'M$ is an ordinate, and B' a point on the curve.

Now $\qquad TB : BT' = B''T : B''T';$ (Prop. XXXVII.)

$\qquad \therefore OB : BO' = OB' : B'O'.$ (Parallel lines.)

PROPOSITION XXXVIII.

If any line cut the asymptotes of a hyperbola in Q, Q', and the curve in P, then the rectangle under PQ and PQ', is equal to the square of the semi-diameter CB' drawn parallel to QQ'.
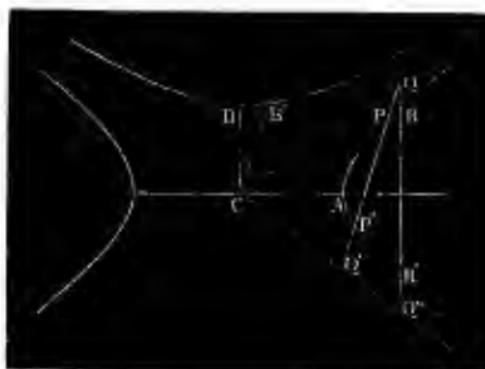

Fig. 44.

Draw the semi-conjugate axis CB. Through Q draw $QRR' \parallel CB$.

Then $\qquad QP \times QP' : QR \times QR' = CB'^2 : CB^2$;

$\qquad\qquad\qquad\qquad\qquad$ (*Cor.* 1, Prop. XXVI.)

but $\qquad QP' = PQ'$, and $QR' = RQ''$; (Prop. XVIII.)

$\qquad \therefore\ QP \times PQ' : QR \times RQ'' = CB'^2 : CB^2$;

but $\qquad QR \times RQ'' = CB^2$; $\qquad\qquad$ (Prop. XVII.)

$\qquad \therefore\ QP \times PQ' = CB'^2$.

Cor.—The portion of any tangent intercepted between the asymptotes is = the parallel diameter.

For $CB'^2 = QP \times PQ' = DE^2$. (See *Cor.* 4, Prop. XVIII., and Fig. 23.)

PROPOSITION XXXIX.

The parallelograms inscribed between the curve and its asymptotes are equal.

Fig. 45.

Let $CMPN$, $CM'P'N'$ be the parallelograms.

Join PP', and produce it to meet the asymptotes in Q, Q.

Then $\square\ CMPN : \square\ CM'P'N' = \begin{Bmatrix} PM : P'M' \\ PN : P'N' \end{Bmatrix}$

(23, VI. Euclid.)

$$= \begin{Bmatrix} PQ : QP' \\ PQ' : P'Q' \end{Bmatrix}.$$

(Similar Δ^s.)

But $PQ = P'Q'$, and $PQ' = QP'$; (Prop. XVIII.)

hence $\square\ CMPN = \square\ CM'P'N'$.

Cor. 1.—The area of the sector of a hyperbola, made by joining any two points on the curve with the centre, is equal to the area included between the curve, one asymptote, and the lines drawn through the points parallel to the other asymptote.

For the $\triangle^s$ $CP'N'$ and CPN are equal in area, being halves of equal parallelograms; take them successively from the figure $CPP'N'$, and sector CPP' = figure $NPP'N'$.

Cor. 2.—The hyperbola continually approaches its asymptotes, but can never meet either at any finite distance.

For, the area of the $\square$ $CMPN$ being constant;

$$\therefore PM \text{ varies inversely as } PN \text{ or } CM;$$

hence, as CM increases, PM will diminish.

And $\therefore$ when CM becomes infinitely great, PM will be infinitely small.

Cor. 3.—The area of the $\triangle$ formed by any tangent to a hyperbola with the asymptotes is constant.

For the portion of a tangent at any point P intercepted between the asymptotes will be bisected at its point of contact (Prop. XVIII.);

$\therefore$ area of $\triangle$ RCS = 2 area of $\square$ $CMPN$, and $\therefore$ constant.

Cor. 4.—If two hyperbolas have the same asymptotes, the segments of any line, drawn parallel to either asymptote, intercepted between the curves and the other asymptote, are in a constant ratio.

If the second hyperbola be supposed to cut PN in p, and $P'N'$ in p';

then $\qquad \square$ CP : $\square$ Cp = $\square$ CP' : $\square$ Cp';

$$\therefore PN : pN = P'N' : pN'.$$

PROPOSITION XL.

If through any point P on a hyperbola lines OPO', QPQ' be drawn parallel to any two adjacent sides AD, DC, of an inscribed quadrilateral, meeting the opposite sides in O, O', and Q, Q': then

$$PO \times PO' : PQ \times PQ' \text{ in a constant ratio.}$$
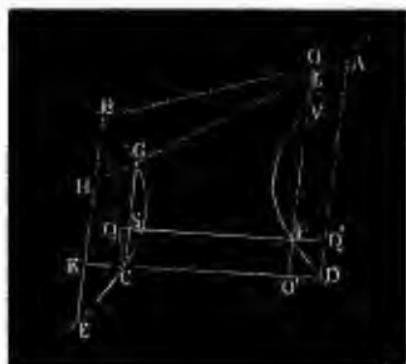
Fig. 46.

Through B and C draw BE and $CG \parallel AD$; join AG, and produce it to meet EB in H.

The diameter which bisects GC and BE will also bisect AD, and will $\therefore$ bisect HK and LO'. Hence $BH = EK$, and $O'V = PL$.

Now $\qquad OL : BH = LA : AH$

$\qquad\qquad\qquad\quad = O'D : DK.$ (Parallel lines.)

Alt. $\qquad OL : O'D = BH : DK.$

Also $\quad PO'$ or $SC : SQ = BK : KC$; (Similar $\triangle^s$.)

$\therefore\ OL \times PO' : O'D \times SQ = BH \times BK : DK \times KC$;

$\therefore\ OL \times PO' : PQ' \times SQ = EK \times BK : DK \times KC$

$\qquad\qquad\qquad\qquad = O'V \times O'P : O'C \times O'D$

$\qquad\qquad\qquad\qquad = PL \times O'P : PS \times PQ'$;

$$\therefore OL \times PO' + PL \times O'P : PQ' \times SQ + PS \times PQ'$$

$$= EK \times BK : DK \times CK ;$$

$$\text{or} \quad PO \times PO' : PQ \times PQ' = EK \times BK : DK \times KC ;$$

Now it is evident that the points A, B, C, D being fixed, E is also fixed, and $\therefore EK \times BK : DK \times KC$ in a constant ratio, and $\therefore PO \times PO' : PQ \times PQ'$ in a constant ratio.

PROPOSITION XLI.

If from any point P on a hyperbola lines PR, PR', PS, PS' be drawn to the sides of an inscribed quadrilateral, making with them any constant angles; then the rectangles under the lines drawn to the opposite sides will be in a constant ratio.
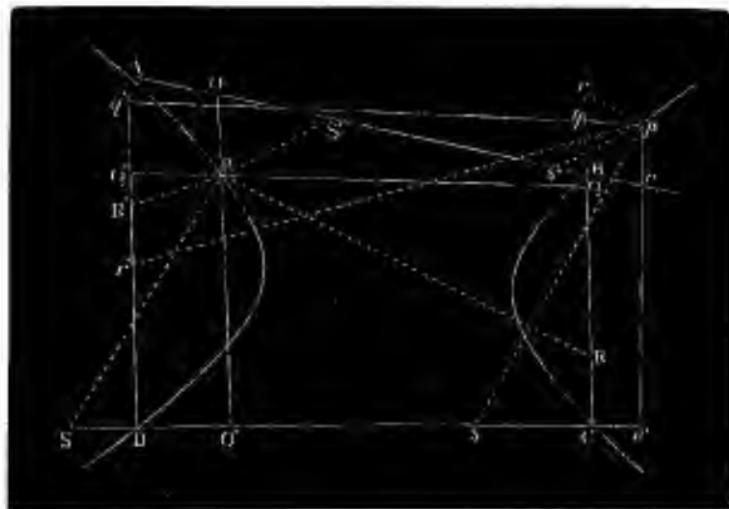

Fig. 47.

Take any other point p on the curve.

Through the points P and p draw QPQ' and $qpq' \parallel DC$, and OPO', $poo' \parallel AD$; also pr, pr', ps, $ps' \parallel PR$, PR', PS, PS' respectively.

Then $\qquad\qquad PR : pr = PQ : pq, \qquad$ (Similar $\triangle^s$.)

and $\qquad\qquad PR' : pr' = PQ' : pq'$;

$\therefore\; PR \times PR' : pr \times pr' = PQ \times PQ' : pq \times pq'.$

Similarly it may be proved that

$$PS \times PS' : ps \times ps' = PO \times PO' : po \times po' ;$$

but $\quad PQ \times PQ' : PO \times PO' = pq \times pq' : po \times po' ;$

$\therefore\; PR \times PR' : PS \times PS' = pr \times pr' : ps \times ps'.$

Cor. 1.—The rectangle under the perpendiculars let fall from any point of a hyperbola on two opposite sides of an inscribed quadrilateral is in a constant ratio to the rectangle under the perpendiculars let fall on the other two sides.

Cor. 2.—If the points A and D coincide, also the points B and C, then the sides AD and BC become tangents, and the sides AB and CD coincide and become the chord of contact. Then the rectangle under the perpendiculars let fall from any point of an hyperbola on two fixed tangents is in a constant ratio to the square of the perpendicular let fall on their chord of contact.

Cor. 3.—If we suppose AB, CD to intersect in X, and AC, BD in Y; also that PR, PR', PS, PS', pr, pr', ps, ps' lie in the lines XY; then, regarding $ACBD$ as the quadrilateral, the above proportion becomes

$$PX^2 : PY^2 = pX^2 : pY^2 ;$$

$\therefore\; XY$ is cut harmonically by the curve.

PROPOSITION XLII.

If two fixed tangents DP, DQ, to a hyperbola be cut by a diameter AB parallel to their chord of contact, and by a third variable tangent EF, the rectangle under the segments of the two fixed tangents intercepted between the diameter and the variable tangent is constant.


Fig. 48.

Join PC; produce it to meet the curve in R; join BR.

Since $CA = CB$; (*Cor.* 2, Prop. xxxv.)

∴ $\triangle^s ACP$ and BCR are equal; (4, I. Euclid.)

∴ BR is $=$ and $\parallel AP$, and ∴ a tangent.

If BR be supposed to be produced to meet FE produced in Z;

then $RB : PE = RZ : PD$ (Prop. xxx.)

$$= RZ - RB : PD - PE = BZ : ED;$$

∴ $AP : PE = BF : FD$;

∴ $AP : AP - PE = BF : BF - FD$;

∴ $AP : AE = BF : BD$;

∴ $AE \times BF = BD \times AP$, and ∴ constant.

Proposition XLIII.

To describe a conic section to pass through five given points.
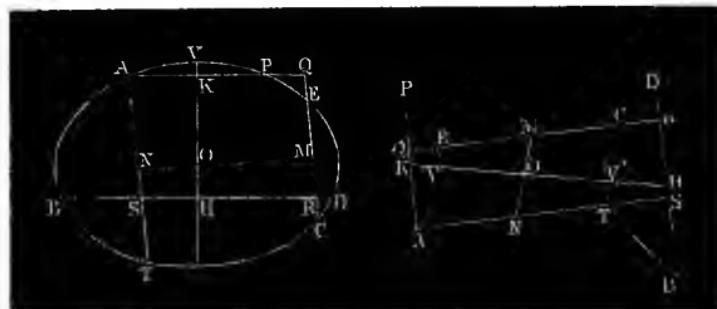
Fig. 49.   Fig. 50.

Join *BD* and *CE*.   Through *A* draw *AQ* ∥ *BD*, and *AS* ∥ *CE*; in *AQ* find a point *P*, and in *AS* a point *T*, such that

$$BR \times RD : CR \times RE = AQ \times QP : CQ \times QE,$$

and $BR \times RD : CR \times RE = BS \times SD : AS \times ST$.

The points *P* and *T* will (Prop. xxii. Chap. ii., and *Cor.* 1, Prop. xxvi. Chap. iii.) be points on the curve.

If ∴ we bisect *AT* in *N*, and *CE* in *M*, and join *MN*, this line will evidently pass through the centre.   Likewise bisect *AP* in *K*, and *BD* in *H*, and join *KH*; this line will also pass through the centre.   Should *MN* and *KH* happen to be parallel the curve will be a parabola ; if not, their point of intersection will determine the centre of the curve.

Now determine *OV*, such that

$$PK^2 : DH^2 = OV^2 \sim OK^2 : OV^2 \sim OH^2;$$

then (*Cor.* 2, Prop. xxii.) *OV* will be a semi-diameter to which *PK* and *DH* are ordinates; hence any number of ordinates and points on the curve may be found.   Also the axes (by Prop. xxx. Chap. ii., and Prop. xxxiv. Chap. iii.).

PROPOSITION XLIV.

To describe a conic section to touch five given right lines.
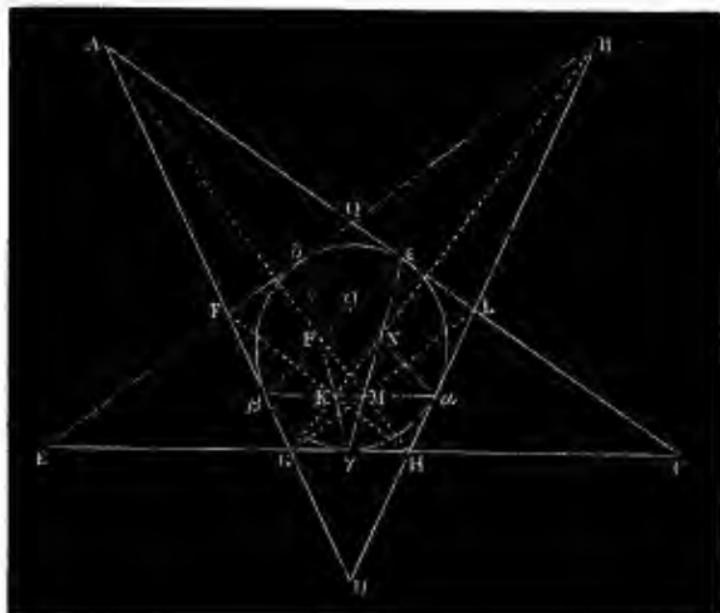
Fig. 51.

First, consider the quadrilateral *BFGH* formed by any four of the five tangents; draw the diagonals; let them intersect in *K*. Next consider the quadrilateral *AGHL* formed by leaving out another tangent; draw the diagonals; let them intersect in *M*.

In this way go round the figure, leaving out in succession each of the tangents, and three other points *N, O, P*, may be similarly found. The line joining *K, M*, produced will intersect the tangents in *a, β*; and the line joining *KP* produced will intersect the tangents in *γ, δ*, points on the curve.

For the intersection K of the diagonals of the quadrilateral $BFGH$ is (by Prop. XL.) a point on the chord of contact of the required conic with the lines FG, LH; also the intersection M of diagonals of the quadrilateral $AGHL$ is, by same Prop., a point on the chord of contact with the same lines; $\therefore$ a, β, are the points of contact of the required conic with these lines. Similarly it may be shown that γ, δ, and ϵ, are the points of contact with the other right lines.

Proposition XLV.

If a right cone be cut by a plane which, when produced, cuts the opposite cone, the section will be a hyperbola.
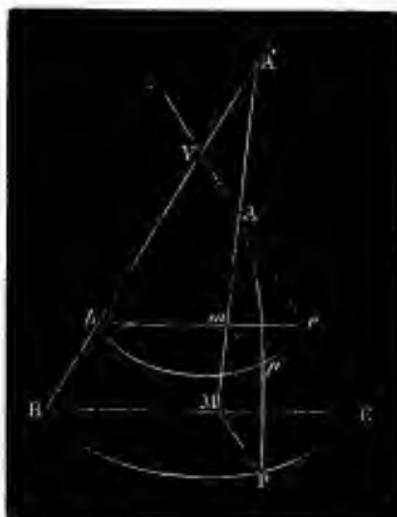

Fig. 52.

Let the plane BVE drawn through the axis of the cone perpendicular to the plane of the section coincide with the plane of the paper, then both the section APM and the base BPE will be $\perp$ the plane of the paper; $\therefore$ the line

MP in which the section cuts the base is $\perp$ the plane of the paper, and $\therefore \perp BE$.

Hence $\qquad BM \times ME = MP^2$. (35, III. Euclid.)

If now any other plane bpe be drawn $\parallel$ the base, meeting the section in pm, it can similarly be shown that mp is $\perp be$, and $\therefore bm \times me = mp^2$.

But $\qquad BM : bm = A'M : A'm$; (Similar $\triangle^s$.)

also $\qquad ME : me = MA : mA$;

$\qquad \therefore BM \times ME : bm \times me = A'M \times MA : A'm \times mA$,

or $\qquad MP^2 : mp^2 = A'M \times MA : A'M \times mA$.

Hence the section is a hyperbola. (Prop. x.)

Cor. 1.—The conjugate axis is a mean proportional between the diameters of the sections drawn through A and A' parallel to the base.

This follows, as in Chap. II., page 108.

Cor. 2.—The spheres inscribed in the cone, to touch the plane of the circle, will determine the foci.

This follows, as in Chap. II., page 109.

APPENDIX.

MANY solutions having been given of Problem 9, page 110, I consider it advisable to add the following :—

Let PP' be the focal chord.

Draw the tangents, and also the normals at P and P'. Suppose the former to intersect in T, and the latter in N. Let the line drawn through $N \parallel$ to the axis intersect PP' in O, and FT in G.

Draw $NH \perp PP'$.

Then T is the centre exscribed to the $\triangle PF'P'$, and N is the centre of the inscribed circle ;

$$\therefore F', N, F, \text{ are in directum} ;$$

also $\qquad TF$ is $\perp PP'$; $\qquad$ (Prop. XIII.)

$$\therefore P'F = PH.$$

(G. & H.'s Euclid, Appendix, Book IV.)

Now $\qquad P' . F'NPT$ is a harmonic pencil ;

$$\therefore F . F'NPT \text{ is a harmonic pencil.}$$

$$\therefore NG, \text{ which is } \parallel F'F \text{ is bisected at } O ;$$

hence $\qquad FO = OH$; $\qquad$ (26, I. Euclid.)

$$\therefore P'O = OP.$$

In *Cors.* 2 and 3, Prop. XI., Chap. III., in order to describe the position of the point, it is assumed that the curve is concave towards the transverse axis. This property is fully demonstrated in *Cor.* 4, Prop. IX. It is to be noted that the essential property of the tangent, proved in Prop. III., does not depend on the assumption.

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